History and Development of Mathematics in India

History and Development of Mathematics in India

Editors

Sita Sundar Ram

Ramakalyani V.

राष्ट्रीय पाण्डुलिपि मिशन
॥ विज्ञानमुपास्व ॥
National Mission for Manuscripts

DK PRINTWORLD
Publishers of Indian Traditions

Cataloging in Publication Data — DK

[Courtesy: D.K. Agencies (P) Ltd. <docinfo@dkagencies.com>]

Annual Conference on History and Development of Mathematics (2018 : Kānchipuram, India)

History and development of mathematics in India / edited by Sita Sundar Ram, Ramakalyani V.

pages cm

Includes texts in Sanskrit (Devanagari and roman).

"The Annual Conference on History and Development of Mathematics 2018 conducted by the Samskrita Academy, Madras, in collaboration with the Mathematics Department of Sri Chandrasekharendra Saraswati Vishwa Mahavidyalaya, Enathur ... 27 to 29 November 2018" – Introduction.

Includes bibliographical references and index.

ISBN 9789380829708

1. Mathematics – India – History – Congresses. 2. Mathematics in literature – Congresses. 3. Sanskrit literature – History and criticism – Congresses. I. Sita Sundar Ram, editor. II. Ramakalyani, V., editor. III. Sanskrit Academy, organizer. IV. Srī Chandrasekharendra Saraswathi Viswa Mahavidyalaya. Department of Mathematics, organizer, host institution. V. National Mission for Manuscripts (India), publisher. VI. Title.

LCC QA27.I4A56 2018 | DDC 510.954 23

ISBN: 978-93-80829-70-8

First published in India, 2022

Editors: Sita Sundar Ram (b.1951); Ramakalyani V. (b.1956)

Published by:
National Mission for Manuscripts
IGNCA, Janpath Building near Western Court
New Delhi - 110001
Phone: (011) 2338 3894; *Fax:* 2307 3340
e-mail: director.namami@nic.in
Website: www.namami.nic.in

Co-published by:
D.K. Printworld (P) Ltd.
Regd. Office: "Vedaśrī", F-395, Sudarshan Park
(Metro Station: ESI Hospital), New Delhi - 110015
Phones: (011) 2545 3975; 2546 6019
e-mail: indology@dkprintworld.com
Website: www.dkprintworld.com

Printed by: D.K. Printworld (P) Ltd., New Delhi.

Foreword

India's contribution to mathematics, spanning from 1200 BCE to 1800 CE is well known. The decimal number system, concept of zero as number and negative numbers were its gifts in addition to its inputs into the fields of arithmetic, algebra and trigonometry. Its classical as well as golden period ranged from fourth to sixteenth century, having contributions come from great scholars like Āryabhaṭa, Varāhamihira, Brahmagupta and Bhāskara II. However, there were scores of mathematicians whose contributions went into oblivion owing to many a reason. Most of their works have not caught the attention of scholars of the last few centuries. This book endeavours to bring to light some of such forgotten mathematicians and their works.

This volume is the proceedings of an Annual Conference on the History and Development of Mathematics, organized by the Samskrita Academy, Chennai in collaboration with the National Mission for Manuscripts (NMM) and the Mathematics Department of Sri Chandrasekharendra Saraswati Vishwa Mahavidyalaya, Enathur under the auspices of the Indian Society for History of Mathematics. It informs us of many manuscripts like *Grahagaṇitapadakam* belonging to *Saurapakṣa*, *Sūryaprakāśa*, a commentary on Bhāskara's *Bījagaṇita*; *Gaṇitāmṛtalaharī* of Rāmakṛṣṇa; a commentary on Bhāskara's *Līlāvatī*, *Gaṇakānanda*; *Karaṇakutūhalasāriṇī* based on the *Karaṇakūtūhala* of Bhāskara II; *Makarandasāriṇī* and *Mahādevīsāriṇī* among many more.

The scholars, who presented papers, unearth many manuscripts of mathematics by unknown authors and delve deep into the contributions of Indian to the different branches of mathematics. It also pays befitting tribute to well-known contemporary mathematicians – T.A. Saraswati Amma and K.S.

Shukla. The former's seminal works cover mathematics in Jain manuscripts and the entire course of geometry in India from the Śulbasūtras to the works of Kerala school. The latter has brought out eleven important works on Indian mathematics and astronomy starting from the *Sūrya Siddhānta* to the *Gaṇita Kaumudī* of Nārāyaṇa Paṇḍita.

It is with immense pleasure that the National Mission for Manuscripts presents this anthology. And it is my hope and belief that this volume will kindle keen interest of young researchers in Indian mathematics and their dedicated efforts will exhume many more unknown works of Indian mathematicians in the days to come.

Pratapanand Jha
Director
National Mission for Manuscripts

Contents

Introduction

THE Annual Conference on the History and Development of Mathematics 2018 conducted by the Samskrita Academy, Chennai, in collaboration with the Mathematics Department of Sri Chandrasekharendra Saraswati Vishwa Mahavidyalaya (SCSVMV), Enathur, under the auspices of the Indian Society for History of Mathematics was held at SCSVMV from 27 to 29 November 2018. It was sponsored by the National Mission for Manuscripts, New Delhi.

The conference was dedicated to the memory of two eminent Indian Historians of Mathematics and Astronomy – Professor T.A Saraswati Amma and Professor K.S. Shukla. This is the centenary year of both these stalwarts of mathematics.

The conference was inaugurated by Shri T.S. Krishna Murthi, former Chief Election Commissioner of India. The function was presided over by Professor Dr Vishnu Potty, Vice-Chancellor, SCSVMV.

In the conference there were forty-six papers in all, covering various branches of mathematics such as arithmetic, algebra, astronomy and geometry both in manuscripts and printed texts. Speakers came from various parts of India. There was also one professor who had come all the way from the University of Switzerland. Here forty-two papers are compiled in this volume. Being the centenary year of these two stalwarts of mathematics, Professor M.D. Srinivas highlighted the seminal contributions of both of them. Professor Saraswati Amma has written many papers covering Jaina mathematics, the use of geometric methods in arithmetic progression and so on. But her magnum opus was *Geometry in Ancient and Medieval India*. It surveys the entire course of development of geometry in India from the Śulbasūtras to the works of the Kerala School. Professor Shukla has brought out landmark editions of eleven important works on Indian mathematics and astronomy starting from *Sūrya Siddhānta* to *Gaṇita Kaumudī* of Nārāyaṇa Paṇḍita.

Manuscripts

There are almanacs belonging to different schools such as Śara, Ārya, Brahmā and Gaṇeśa. The *pratibhāgī* tables are very popular among the *pañcāṅga* makers of Karnataka and Andhra regions. In her paper, K. Rupa has discussed some features of *pratibhāgīgaṇita* (*PRB*) and *tyāgarti* manuscript *Grahagaṇitapadakāni* belonging to the *saura-pakṣa*. A comparison of parameters in these tables among themselves as also with modern is attempted.

Sita Sundar Ram in her paper on the manuscript of *Sūryaprakāśa*, a commentary on Bhāskara's *Bījagaṇita*, has analysed the text from various angles to highlight the contribution of Sūryadāsa. It is evident that Sūryadāsa is not only a mathematician but also a versatile poet.

Recently, the critical edition of the commentary *Gaṇitāmṛtalaharī* of Ramakrishna on Bhāskara's *Līlāvatī* has been taken up as a project under the National Mission for Manuscripts by Ramakalyani. The date of the text has been differently noted in the colophon and the *New Catalogus Catalogorum*. The date is to be fixed based on other evidences. The author has discussed some important features noticed in the manuscripts.

The text *Gaṇakānanda* is very popular among the *pañcāṅga* makers of the *saura-pakṣa* in Andhra and Karnataka regions. Padmaja Venugopal has edited the text and given valuable English expositions. Her work is based on a single edited text in Telugu script.

Eclipses are natural phenomena which played an important role in the religious life of ancient India. They occupied a significant place in the astronomical texts. The *Grahaṇamālā* of Hemāṅgadaṭhakkura is one such text which lists many eclipses occurring between 1620 and 1708 CE. Vanaja, Shailaja and S.

Balachandra Rao have critically studied the text and compared many results therein with modern ones.

The tables of *Karaṇakutūhalasāriṇī* which relies on the *Karaṇakutūhala* of Bhāskara II of twelfth century are based on *brahma-pakṣa*. The author and time of the text are not known though manuscripts of the same are available in many libraries. M. Shailaja has obtained with rationale, the mathematical model for the construction of tables.

Among the tables of *saura-pakṣa* which are used for compiling the *pañcāṅga*s, the *Makarandāsāriṇī* is very popular. These tables with explanatory *śloka*s are composed by Makaranda at Kāśī in 1478. As noted by S.K. Uma, this text has some unique features such as determination of *ahargaṇa* in the sexagesimal system.

The significance of mathematics and its applications was well realized in ancient times. Consequently, there are a number of texts on the subject. There are very renowned national libraries which have a number of manuscripts on mathematics both published and unpublished. Bhuvaneswari takes three of these in Tamil Nadu and discusses the manuscripts available, scope for critical edition and future research.

Astronomical tables known as *sāriṇī* are usually a collection of necessary data and rules for standard astronomical data. Shubha along with B.S. Shylaja and P. Vinay has studied the manuscript *Mahādevīsāriṇī* and compared the positions of certain planets with modern calculations.

Algebra

Long before the time of Fibonacci (twelfth century) the sequence was known in India. It was applied in connection with metrical science by Piṅgala, Hemacandra and others. The concept of Fibonacci numbers was more advanced in the *Gaṇitakaumudī* of Nārāyaṇa Paṇḍita. Vinod Mishra has discussed the development of Fibonacci sequence and possible applications in statistics, coding theory, medicine and others.

The definition given by various ancient Indian mathematicians for the term *karaṇī* matches with the modern mathematical term surd. Śulbasūtras deal with the rules and measurements for constructing the fire altar, where the words *dvikaraṇī* and *trikaraṇī* occur. They were denoted by *ka* 2, *ka* 3 and so on. Padmapriya has discussed the treatment of surds in ancient mathematical texts.

Bhāskarācārya in his *Bījagaṇita* has devoted a whole chapter to surds or square roots of irrational numbers. S.A. Katre in his paper has analysed the methods employed by Bhāskara for finding the square roots of quadratic surds.

A fore-shadowing of Banach's fixed point theorem appears in the iterative methods of Indian mathematicians from at least the sixth century. The Indian manuscript tradition does not contain drawings but Habash Al-Hasib's contains some drawings though not for the iterative methods. Johannes Thomann has discussed some possibilities through a reading of the related texts.

Arithmetic

The topic of arithmetic progression has been in Indian mathematics for a long period and every mathematician has dealt with it. The aim of Medha Limaye has been to compare and contrast the method of exposing the concept and developing solution techniques in the medieval and modern texts. The geometric representation of the arithmetic progression series by Sridhara and its link to recreational mathematics by Nārāyaṇa Paṇḍita are also discussed.

The *Līlāvatī* of Bhāskara is perhaps the most popular text in mathematics which has been critically edited by various mathematicians in different languages all over the world. Bapudeva Sastri, a reputed mathematician of the nineteenth century, is one of those who has critically edited the text with new *sūtra*s and explanations. In her paper, Vijayalakshmi has thrown light on some of his techniques and examples.

The *Pañcaviṁśaṭīkā* and the *Parikarmacatuṣṭaya* are two texts of unknown authorship which have been recently edited by well-known mathematician and Indologist Takao Hayashi. The contents of the texts dealing with the basic mathematical operations are studied by Umamahesh in his paper.

Astronomy

Parameswara belongs to the Kerala School of Astronomy and Mathematics and is the author of several texts including *Dṛggaṇita.* The text *Vākyakaraṇa* of Parameswara is unique and Venkateswara Pai has taken for analysis and explanation some algorithms in obtaining the *vākya*s.

The observational aspects of Indian astronomers are covered in almost all texts. Measurements had to be accurate and the division of *aṅgula* for example into *vyaṅgula* is noticed in several texts. Hence new words were coined for the need. B.S. Shylaja has made a list of such new words and discusses them.

The infinite series expansion for the sine and cosine functions is generally ascribed to Mādhava of Saṅgamagrāma. There is a short work *Mahājyā* which describes the infinite series for the *jyā* $R \sin \theta$ and *sara* $\{R(1 - \cos \theta)\}$. Rajarajeswari worked on a manuscript of this for her MPhil thesis submitted to the University of Madras. She has translated the work into English. In the present paper she has explained some derivations.

It was the standard Indian practice to revise the parameters associated with the sun and the moon after critically testing them during eclipses. Hannah Thangam has discussed a simplified version of the calculations pertaining to lunar eclipses in the *Tantrasaṁgraha.* For some recent lunar eclipses there was very good agreement with the values computed using *Tantrasaṁgraha* and the tabulated values.

For the mathematicians and astronomers of India, the *trairāśika* (rule of three) and the theorem of the right angle play a crucial part in the derivation of all the results related to the planetary positions. To substantiate, M.S. Sriram considered some examples from Bhāskara's *Grahagaṇita*, one of them being the derivation of a second-order interpolation formula due to the renowned Brahmagupta of the seventh century.

Indian mathematics encompasses the era of Kerala School of Mathematics which has been reckoned as the golden age in the history of mathematics. Starting from texts of Mādhava of

Saṅgamagrāma, Anil Narayanan traced the lineage of Kerala tradition. He has stressed the feature of continuity in the tradition by taking an annual called *Śuddhadṛgganita* into account.

The only paper in Sanskrit was presented by S. Murali. In this he has stressed the importance of *kāla* (timing) to perform rituals. He has explained briefly the *kālanirṇaya* for both social and individual rituals.

Geometry

The *Mānava Śulbasūtras*, one of the four major Śulbasūtras of significance in terms of mathematical contents, has received relatively less attention compared to the *Baudhāyana*, *Āpastamba Śulbasūtras*. S.G. Dani discussed various general features and certain unique constructions from the *Mānava Śulbasūtras*, placing it in the overall context of Śulba literature.

The major part of mathematical principles was passed on from generation to generation and some of them have been recorded orally in *sūtra* forms. The Śulbasūtras depict major theorems in modern geometry. Sudhakar Agarkar has highlighted the geometrical knowledge of ancient Indians as presented in Śulbasūtras.

The important branch of mathematics which received most attention was geometry. Most civilizations had detailed texts on the subject. In his paper, Shrenik Bendi has explored how geometry was developed and discussed various results obtained by Vedic and Jaina scholars.

In his paper on T.A. Saraswati Amma, Chandrasekharan has analysed the methods used by her in some important topics such as segments of circles, cyclic quadrilaterals, trigonometric and inverse trigonometric series. He has also suggested extending her work on areas where she has only touched upon briefly due to paucity of time.

General

A unique website https: indiamathstory.com was presented by Sarada Devi. The website covers many mathematics conferences, details of reference books, resource persons, questions–answers, riddles and so on. It is indeed a continuous saga as it has room for various additions in the future.

The Vedas are the oldest unrecorded transmission of sound. The Vedic *mantra*s had to be heard from a *guru* and memorized along with the sound. There are eight *vikṛti*s for chanting the Vedas. Of these, the *gaṇa pāda* is very complex. The *mantra*s classified by Sage Vyāsa serve as an astro-chronological computation methodology. This technology when mastered and adapted can greatly enhance the knowledge transmission. Rajendran has listed those which are useful for current technology.

Modern Themes

The following are various papers in modern mathematics presented by the staff and students of the Mathematics Department of Sri Chandrasekharendra Saraswati Vishwa Mahavidyalaya, Enathur.

- In her paper "A Note on Confusion Matrix and Its Real Life Application", T.N. Kavitha has discussed the origin of the confusion matrix and definitions of various persons are given in a detailed manner.
- In the nineteenth century, hydrodynamics advanced sufficiently to derive the equation of motion of a viscous fluid by Navier and Stokes, only a laminar flow between parallel plates was solved. In the present age, with the progress in computers and numerical techniques in hydrodynamics, it is now possible to obtain numerical solutions of Navier–Stokes equation. E. Geeta and M. Larani discuss this in "Historical Development of Fluid Dynamics".
- A. Dhanalakshmi and K. Srinivasa Rao have reviewed the Hosoya Polinomial and Weiner Index and some of the methodology used in it so far in "Role of Weiner Index in Chemical Graph Theory".
- In the paper "The Origin of Semiring-valued Graph", Ramya and T.N. Kavitha discuss the origin of Semiring-valued graph and its application fields.

- In "History of Optimization Models in Evolutionary Algorithms", K. Bharathi has discussed about the history of the framework, related algorithms developed and their applications and some of the methodologies used in it.
- Graph theory has uses beyond simple problem formulation. Sometimes a part of a large problem corresponds exactly to a graph-theoretic problem, and that problem can be completely solved. C. Yamuna and T.N. Kavitha analyse this "Graph Used to Find Crime".
- B. Akila presents the application of calculus in the transition curve for a rail track in her paper "A Discussion on Real-Life Application of Mathematics".
- Operations research includes a great deal of problem-solving techniques like mathematical models, statistics algorithms to aid in decision making. J. Sengamalaselvi, in her paper "History of Operations Research" traces the history of Operations Research.
- The machine learning methods analyses and extracts knowledge from available data and provides an easier way to understand the graph structured data: proteins, protein–protein interaction, protein structures along chemical pathways, social networks, WorldWideWeb, Program flow. The prime objective of Vijayalakshmi, in her paper "Graph Kernels in Protein Study" is to present a survey of graph kernels in protein study.
- In "Spectral Techniques in Protein Study: A Survey", K. Divya and Vijayalakshmi give a survey of graph spectral techniques used in protein study. This survey consists of description of methods of graph spectra used in different study areas of protein like protein domain decomposition, protein function prediction and similarity.
- "Review of Weiner Index and Its Applications" the aim of this article is to review the history of the Wiener Index and its progress achieved in the recent years. V. Kasthuri and A. Dhanalakshmi have reviewed the development of the index and its applications in various fields.

- In "MATLAB in Protein Study", D. Vijayalakshmi and A. Shakila brief about the use of MATLAB in various studies of proteins encode, amount of protein adsorption on particle, sequence alignments, protein structure tessellations which help in making the studies easy.

The Valedictory Function of the conference was held on 29 November 2018. The Valedictory Address was delivered by Dr V. Kannan, Former Pro-Vice-Chancellor, University of Hyderabad.

The conference ended on a happy note with the Blessings of His Holiness Sankara Vijayendra Sarasvati Svamiji of Kāñcī Mutt and with assurances to meet again in the next Conference on History of Mathematics.

We are deeply beholden to many people including the scholars who have presented the articles, in the successful conduct of the conference. In this connection our special thanks are due to Dr Pratapananda Jha, Director and Dr Sangamitra Basu, Coordinator, Publications of the National Mission for Manuscripts, who stood by us from the beginning of the conference till the publication of its proceedings. We are also deeply indebted to Dr M.K. Tamil Selvi (Associate Professor, Alpha College of Engineering, Chennai), for reviewing all the papers on modern topics in mathematics.

Dr. Sita Sundar Ram
(Secretary, The Samskrita Academy, Chennai)

Dr. V. Ramakalyani
(Project Consultant, HoMI Project, IIT, Gandhinagar & Member,
The Samskrita Academy, Chennai)

1

Tribute to T.A. Saraswati Amma and K.S. Shukla[1]

M.D. Srinivas

T.A. Saraswati Amma (1918–2000)

Tekkath Amayankoth Kalam Saraswati Amma was born in Cherpulassery, Palakkad, Kerala, in the Kollam year 1094 (1918-19). She completed her BSc with Physics and Mathematics from Madras University, and Masters degree in Sanskrit from Banaras Hindu University. In 1957, she joined the Department of Sanskrit, Madras University, and worked with the renowned Sanskrit scholar V. Raghavan, who encouraged her to work on the history of Indian Mathematics. In 1961, Saraswati Amma joined the faculty of Department of Sanskrit, Ranchi Women's College, where she worked for the next twelve years. In 1964, Saraswati Amma was awarded the PhD degree by the Ranchi University for her

T.A. Saraswati Amma

[1] Excerpts from the talk given by Prof. M.D. Srinivas: "Recollecting the Seminal Contributions of T.A. Saraswati Amma and K.S. Shukla", at the Annual Conference on History and Development of Mathematics, 2018. – *Editors*

dissertation (which was Series submitted in January 1963) on "Geometry in Ancient and Medieval India". While at Ranchi, she also supervised the thesis of R.C. Gupta on "Trigonometry in Ancient and Medieval India". During 1973-80, Saraswati Amma served as Principal, L.N.T. Mahila Vidyalaya, Dhanbad. After retirement, Saraswati Amma returned to Kerala and stayed in Ernakulam and later at Ottappalam.

Publications of Saraswati Amma

1. "Śreḍhī-kṣetras or Diagrammatic Representation of Mathematical Series", *Journal of Oriental Research*, 28 (1958-59): 74-85.
2. "The Cyclic Quadrilateral in Indian Mathematics", *Proceedings of the All-India Oriental Conference*, 21 (1961): 295-310.
3. "The Mathematics of the First Four Mahādhikāras of the Trilokaprajñapti", *J. of the Ganganath Jha Research Inst.*, 18 (1961-62): 27-51.
4. "Mahāvīra's Treatment of Series", *Journal of Ranchi University*, I (1962): 39-50.
5. "The Development of Mathematical Series in India after Bhāskara II", *Bulletin of the National Inst. of Sciences of India*, No. 21 (1963): 320-43.
6. "Development of Mathematical Ideas in India", *Indian Journal of History of Science*, 4 (1069): 59-78.
7. "Sanskrit and Mathematics", Paper read at the First International Sanskrit Conference, New Delhi, 1972, see Summary of Papers, vol. III, pp. 15-16, published in the Proceedings (of the conference), vol. III, part I, pp. 196-200 (New Delhi, 1980) Also reprinted in the "Souvenir of the World Sanskrit Conference, New Delhi, 2001, 63-78"
8. "The Treatment of Geometrical Progressions in India", Paper sent to Indian Science Congress for Symposium on History of Mathematics (Delhi, 1975). For a summary, see *Advance Notes on Symposia*, pp. 7-8.

9. "Indian Methods of Calculating the Volume of the Frustrum of a Pyramid", in *Sanskrit and Indology*: *Dr. V. Raghavan Felicitation Volume*, Delhi, pp. 335-39, 1975.
10. "Bhāskarācārya", in *Cultural Leaders of Indian Scientists,* ed. Raghavan, New Delhi, 1976, pp. 100-106.
11. Reviews of *Candracchāyāgaṇita*, *Siddhāntadarpaṇa* and *Sphuṭanirṇaya-tantra*, *Vishveshvaranand Indological J.*, 15 (1977): 173-76.
12. *Geometry in Ancient and Medieval India*, Delhi, 1979; xii + 280 pp., rev. 2nd edn, Delhi 1999. The book was reviewed by S. Balacandra Rao in *Deccan Herald Magazine* dated 21 October 1979; by A.K. Bag in *Gaṇita Bhāratī*, vol. 3 (1981), pp. 53-54; by Michio Yano in *Historia Mathematica* 10 (1983): 467-70; by A.I. Volodarsky in *Mathematical Reviews* 84 (1984): 2516-17; by D.G. Dhavale in the *Annals of Bhandarkar Oriental Research Institute*, vol. 69 (Pune, 1988) and by J.N. Kapur in *JHS*, 24 (1989): 93-94.
13. Review (by T.A. Saraswati Amma) of *Geometry according to Śulabasūtra* (authored by R.P. Kulkarni, Pune, 1983), *Gaṇita Bhāratī,* 8 (1986): 64-65. See vol. II (1989): 60-62 for Kulkarni's reply to the review.

Saraswati Amma's magnum opus, however, is her book *Geometry in Ancient and Medieval India*. Saraswati Amma's book has been widely acclaimed as a worthy successor to the volumes of Datta and Singh, as it presents a truly majestic survey of the entire course of development of Geometry in India, from the Śulbasūtras to the work of the Kerala School. Saraswati Amma has also taken great pains to present original citations and translations of important verses, both from published works as well as unpublished manuscripts. Some of the works cited by her, such as the commentary of Parameśvara on *Līlāvatī*, are yet to see the light of the day. Saraswati Amma's book still constitutes the standard reference for students on Indian Geometry.

K.S. Shukla (1918–2007)

Kripa Shankar Shukla was born on 12 June 1918 in Lucknow. He completed his undergraduate and postgraduate studies in Mathematics at Allahabad University. In 1941, Shukla joined the Department of Mathematics, Lucknow University, to work with Prof. Avadhesh Narayan Singh (1905-54). Professor Singh, the renowned collaborator of Bibhutibhusan Datta (1888–1958), had joined Lucknow University in 1928.


K.S. Shukla

Shukla's first paper, published in 1945, presented a clear and comprehensive survey of the second correction (due to evection) for the Moon. In 1955, Shukla was awarded the D Litt degree from Lucknow University for his thesis on "Astronomy in the Seventh-century India: Bhāskara I and His Works". Dr. Shukla became the worthy successor of Professor Singh to lead the research programme on Indian Astronomy and Mathematics at Lucknow University. Though he retired as Professor of Mathematics in 1979, he continued to guide researchers and work relentlessly to publish a number of outstanding articles and books, including an edition and translation of *Vaṭeśvarasiddhānta* (*c*.904), the largest known Indian astronomical work with over 1,400 verses, brought out by INSA in 1985-86.

Professor Shukla wrote popular textbooks on Trigonometry (1951) and Algebra (1957). He also published Hindi translations of the first volume of *History of Hindu Mathematics* by B.B. Datta and A.N. Singh (in 1956), and the textbook on Calculus by A.N. Upadhyay (1980).

Professor Shukla's Editions and Translations of Source-Works of Indian Astronomy and Mathematics

Professor Shukla brought out landmark editions of eleven important source-works of Indian Astronomy and Mathematics.

The books edited by K.S. Shukla and published by Lucknow University, Lucknow:

1. *Sūryasiddhānta* with commentary of Parameśvara (1957).
2. *Pāṭīgaṇita* of Śrīdharācārya, ed. and tr. with Notes (1959).
3. *Mahābhāskarīya* of Bhāskara I, ed. and tr. with Notes (1960).
4. *Laghubhāskarīya* of Bhāskara I, ed. and tr. with Notes (1963).
5. *Dhīkoṭidakaraṇa* of Śrīpati, ed. and tr. with Notes, Akhila Bharatiya Sanskrit Parishad, Lucknow (1969).
6. *Karaṇaratna* of Devācārya, ed. and tr. with Notes (1979).
7. *Bījagaṇitāvataṁśa* of Nārāyaṇa Paṇḍita, ed., Akhila Bharatiya Sanskrit Parishad, Lucknow (1970).

The following books are edited by K.S. Shukla and published by Indian National Science Academy, New Delhi:

8. *Āryabhaṭīya* of Āryabhaṭa, ed. and tr. with Notes, with by K.V. Sarma (1976).
9. *Āryabhaṭīya* of Āryabhaṭa with the commentary of Bhāskara I (1976).
10. *Vaṭeśvarasiddhānta* of Vaṭeśvara, ed. and tr. with Notes, 2 vols. (1985-86).
11. *Laghumānasa* of Mañjula, ed. and tr. with Notes (1990).
12. Professor Shukla also collaborated with renowned scholar Samarendra Nath Sen (1918-92), in editing the pioneering *History of Indian Astronomy* brought out by the Indian National Science Academy in 1985 (2nd edn published in 2000).

Professor Shukla also wrote over forty important articles, which have ushered in an entirely new perspective on the historiography of Indian Astronomy and Mathematics. In 1954, Shukla published an article on "Ācārya Jayadeva: The Mathematician" where he brought to light the verses of Jayadeva on *vargaprakṛti* and the *cakravāla* method, as cited in a manuscript of the commentary *Sundarī* of Udayadivākara (*c*.1073) on *Laghubhāskarīya*. This commentary still remains unpublished.

2

A Comparative Study of Pratibhāgī Gaṇitam and Tyāgarti Manuscript Grahagaṇita-Padakāni

K. Rupa
Padmaja Venugopal
S.K. Uma
S. Balachandra Rao

Abstract: Compilers of annual calendrical-cum-astronomical almanacs (*pañcāṅga*s) depend on traditional astronomical tables called differently as *sāriṇī*s, *padaka*s, *vākya*s and *koṣṭaka*s. There are a large number of such tables belonging to different schools (*pakṣa*s) like Saura, Ārya, Brāhma and Gaṇeśa.

In the present paper we discuss some features of *Pratibhāgī Gaṇitam* (*PRB*) and Tyāgarti manuscript *Grahagaṇitapadakāni* belonging to the *saura-pakṣa*. A comparison of parameters in these tables among themselves as also with modern is attempted.

Keywords: Astronomical tables, *pañcāṅga*s, *pratibhāgī*, *Grahagaṇitapadakāni*.

Introduction

THE *Pratibhāgī Gaṇitam*[1] tables are very popular among the *pañcāṅga*

[1] A copy of the *Pratibhāgī Gaṇitam* (*PRB*) manuscript procured from the Oriental Research Institute (ORI), Mysore.

makers in Karnataka and Andhra regions. Most possibly the name of the text comes from the fact that the relevant tables are computed for each degree (*pratibhāga* in Kannada). *Pratibhāgī* in contrast to the Siddhānta and Karaṇa texts provides tables for each degree.

The *Grahagaṇitapadakāni,* this manuscript belongs to a small place called Tyāgarti (also Tāgarti) of Sagar *tāluka* in Shimoga district of Karnataka. This manuscript is based on the *Sūrya-Siddhānta.*

Pratibhāgī Gaṇitam

The *Pratibhāgī Gaṇitam* tables are very popular among the *pañcāṅga* makers in Karnataka and Andhra regions. Most possibly the name of the text comes from the fact that the relevant tables were computed for each degree (*pratibhāga*).

Āryabhata I (b.476 CE) and the now popular *Sūrya-Siddhānta* provide R sin differences (R = 3438') to get R sin for every 3°45'. Some Karaṇa texts (handbooks) provide brief tables for the *manda* and *śīghra* equations for the respective anomalies at even higher interval (step-lengths). For example, Gaṇeśa Daivajña in his *Grahalāghavam* (1520 CE) tabulates the *manda* and *śīghra* equations of the planets at intervals of 15°. Another popular handbook, the *Karaṇakutūhalam* of Bhāskara II (b.1114 CE) gives the *jyā khaṇḍa*s (blocks of R sin values) for every 10°. In such cases intermediate values are obtained by interpolation.

Now, the *Pratibhāgī Gaṇitam* in contrast to the Siddhānta and Karaṇa texts provides tables for each degree. In the photocopy with us, no mention of either the author or of the period of the composition is mentioned. The mean positions of the heavenly bodies have to be worked out using the Kali *ahargaṇa,* the elapsed number of civil days for the given date from the beginning of the Kali-Yuga (the mean midnight between 17 and 18 February 3102 BCE). Therefore, the *Pratibhāgī Gaṇitam* text has no need to mention or use a later epoch.

The popularity of the *Pratibhāgī Gaṇitam* in parts of Karnataka and Andhra regions is very clear from the fact that a good number of manuscripts of the main text as also its commentaries are listed in the Catalogue of ORI, Mysore.

fig. 2.1: First page of the *Pratibhāgī Gaṇitam*

The important table in *Pratibhāgī Gaṇitam* are on:

1. the mean motions of the sun, the moon, apogee (*mandocca*) and the ascending node (Rāhu) of the moon and the five planets;
2. the *mandaphala* (equation of centre) of the bodies;
3. the *śīghraphala* (equation of conjunction) of each planet;
4. the sun's declination (*krānti*); and
5. the moon's latitude (*vikṣepa*, *śara*).

The tables of mean motions of the bodies for each day from 1 to 9 days, every 10 days from 10 to 90 days, every 100 (*nūru* in Kannada) days from 1 to 9 hundreds, every 1,000 (*sāvira* in Kannada) from 1,000 to 9,000, from 10,000 to 9,000, 1 to 9 lakh (hundred thousand, *lakṣa* in Sanskrit and Kannada) and finally for 10 and 20 lakh (i.e. one and two million) days.

Mean Motion, Revolutions and Sidereal Periods in the Pratibhāgī Gaṇitam

From the mean motion of the sun for two million days given in the *Pratibhāgī Gaṇitam*, we have $5475^{\text{Rev.}}\ 6^{S}25°18'33''02'''$ (the superscript S stands for "signs", i.e. *rāśi*s of the zodiac). This gives us the sun's mean daily motion, $SDM = 0°.985602617263794$. From SDM, we obtain the length of the *nirayaṇa* (sidereal) solar year = 365.2587703139661 days and *sāvana-dina*s (civil days) in a *mahāyuga* (of 432×10^4 years) as 1,577,917,888 days.

The number of civil days in a *mahāyuga* according to the *Sūrya-Siddhānta* is 1,577,917,828 so that the *bīja* (correction) for civil days is +60.

We list the mean daily motions, revolutions (*bhagṇa*s) and the sidereal periods of the bodies according to *Pratibhāgī Gaṇitam* in Table 2.1.

***Note*:** In Table 2.1, (i) the mean daily motions are given correct to 15 decimal precision (on computer), (ii) the revolutions in a *mahāyuga* (of 432×10^4 solar years) are given to the nearest integer, and (iii) the sidereal periods are correct to 4 or 5 decimal places.

Tyāgarti Manuscript Grahagaṇitapadakāni

We procured recently a copy of a manuscript, called *Grahagaṇitapadakāni*[2] from a private collection. The manuscript belongs to a small place called Tyāgarti (also Tāgarti) of Sagar

Table 2.1: Daily Motion, Revolutions and Sidereal Periods in Pratibhāgī Gaṇitam

Body	*Mean Daily Motion*	*Revolutions in Mahāyuga*	*Sidereal Period*
Moon	13°.17635250091553	57,753,339	27.32167
Moon's *mandocca*	0°.1113829091191292	488,203	3232.0937
Rāhu	0°.0529848113656044	232,238	6794.4
Kuja	0°.5240193605422974	2,296,832	686.9975
Budha's *śīghrocca*	4°.092318058013916	17,937,061	87.9697
Guru	0°.08309634029865265	364,220	4332.32076
Śukra's *śīghrocca*	1°.60214638710022	7,022,376	224.69857
Śani	0°.03343930840492249	146,568	10765.7729

[2] The Tyāgarti manuscript was procured by the present authors from Dr Jagadish of Shimoga.

tāluka in Shimoga district of Karnataka. The latitude (*akṣa*) of the place is given in terms of *akṣabhā* (*palabhā*). This value coincides closely with the known modern value of the latitude of Tyāgarti.

The *Grahagaṇitapadakāni* explicitly mentions that it is based on the *Sūrya-Siddhānta.* Even like the *Pratibhāgī Gaṇitam*, the *Grahagaṇitapadakāni* does not need and does not mention a contemporary epoch. Both of them need the Kali *ahargaṇa* for a given date. Kali *ahargaṇa* (*KA*) represents the number of civil days elapsed since the beginning of the Kali-Yuga, viz. the mean midnight between 17 and 18 February 3102 BCE.

This Kali *ahargaṇa* accumulated to more than ten lakh (one million) days around 365 BCE. For example, as on 1 August 2011, *KA* = 1,867,309, more than 1.8 million days. Therefore, both the *Pratibhāgī Gaṇitam* and the *Grahagaṇitapadakāni* manuscripts provide the mean motion tables even for a lakh, ten lakh and a crore (ten million) days for the sake of accuracy. These data help us to obtain the sidereal period and the *bhagṇa*s (revolutions in a *mahāyuga*) of a heavenly body.

The *Grahagaṇitapadakāni* contains 32 folios of tables for astronomical computations. One or two folios are missing in between. For example, the folio for the mean motion of Saturn (*śanimadhyapadakāni*) is missing in the bundle of folios.

Interestingly, the manuscript is in Nāgarī script with numerals completely in Kannada script. Even many Kannada words, by

fig. 2.2: Folio from Tyāgarti manuscript the way of instructions or descriptions, are in the Nāgarī script. Folio 31 (back) mentions *akṣaliptāḥ* 842 | 17", i.e. the latitude in arc minutes is 842 | 17. This means the local latitude ϕ = 842'17" = 14°02'17". Further, folio 32 mentions *laṅkodayaviṣuvacchāyāṅgula* 3. This means that the equinoctial shadow (called *akṣabhā* or *palabhā*) is 3 *aṅgula*s (with the gnomon of length 12 *aṅgula*s). This gives:

$$\text{latitude}, \varphi = \tan^{-1}\left(3/12\right) = \tan^{-1}(0.25) = 14°02'10''.48$$

Folio 11 (front) mentions *kalivarṣa* 4813. Now, Kali year 4813 corresponds to 1712 CE. In the same folio the *mandocca*s (apogees) and the *pāta*s (nodes) of the planets are given.

Although for obtaining the mean positions, contemporary epoch is not needed, the author of the *Grahagaṇitapadakāni* perhaps desired updation of the apogee and nodes of the planets. However, the rates of motion of these special points as given in the *Sūrya-Siddhānta* are unrealistic from the point of view of our modern known results.

In addition to giving the Kali year as 4813 (1712 CE), *Grahagaṇitapadakāni* mentions the *nirayaṇa* mean position of the sun as $11^{Ra}10°08'03''$ which gives the date as 22 March of the year 1712 CE with *ayanāṁśa* (amount of equinoctial precession) as about 18°. From this data the *Grahagaṇitapadakāni* can be dated as 22 March 1712 CE, three centuries old.

Solar Year, Civil Days, Revolutions, etc. in Grahagaṇitapadakāni

The *Grahagaṇitapadakāni* gives the sun's mean motion for 1 crore (10^7) days as $10^{Ra}06°33'20''$ (along with 27,377 revolutions as can be calculated). From this we get

(i) The sun's mean daily motion, *SDM* = 0°.9852676868.

Therefore, in a *mahāyuga* of 4,320,000 solar years, the number of civil days (*sāvana-dina*s):

$$\frac{4{,}320{,}000 \times 360°}{SDM} = 1{,}577{,}917{,}792$$

Table 2.3: Mean Daily Motions, Revolutions and Bījas in Grahagaṇitapadakāni

Body	*Mean Motion for 1 Crore Days*					*Revolutions in Mahāyuga*		*Bīja*
	Revolution	*Ra*	*D*	*M*	*S*	*TYGMS*	*SS*	
Moon	366,009	9	11	27	8	57,753,332	57,753,336	–4
Moon's *mandocca*	3,093	11	19	6	20	488,202	488,203	–1
Rāhu	1,471	9	18	8	0	232,237	232,238	–1
Kuja	14,556	1	3	46	40	2,296,832	2,296,832	0
Budha's *śīghrocca*	113,675	6	0	26	30	17,937,059	17,937,060	–1
Guru	2,308	2	23	25	20	364,219	364,220	–1
Śukra's *śīghrocca*	44,504	0	23	56	0	7,022,375	7,022,376	–1

The corresponding value according to the *Sūrya-Siddhānta* is 1,577,917,828. Therefore, *bīja* (correction) of civil days is – 36 and

(ii) the length of the *nirayaṇa* solar year = $360°/SDM$ = 365.2587563 days.

Based on the mean motions of the bodies for ten million days in the *Grahagaṇitapadakāni*, we have worked out *bhagṇa*s (revolutions) and hence the *bīja* as shown in Table 2.3.

In Table 2.3 we observe:

i. the mean motions are given for one crore (10 million) days in terms of revolutions, *rāśi*s (signs), degrees (*aṁśa*), minutes (*kalā*s) and seconds (*vikalā*s),

ii. revolutions in a *mahāyuga* are to the nearest integer,

iii. the last column gives the *bīja*s (correction) to the revolutions given in the *Sūrya-Siddhānta*, and

iv. details of Śani do not appear in the table since the related folio is missing in the *Grahagaṇitapadakāni*.

Mandaphalas and Śīghraphalas in Pratigāmī Gaṇitam and Grahagaṇitapadakāni

fig. 2.3: Folio from the *Pratigāmī Gaṇitam* consisting of Śukra's *mandaphala* and *śīghraphala*

In finding the true longitudes of the sun and the moon we need apply only the major correction, *mandaphala* (equation of centre). But, in the case of the five planets, besides the *mandaphala*, the other major equation to be applied is *śīghraphala*s.

Mandaphala in the Saura Tables

The *mandapala* (equation of centre) of a heavenly body is given by the classical expression:

$$\sin(MP) = \frac{P}{R}\sin(MK), \tag{1}$$

where MP is the required *mandaphala*, MK is the *mandakendra* (anomaly from the apogee), p is the *mandaparidhi* (periphery of the related epicycle), $R = 360°$, the periphery of the deferant circle. The *mandakendra* is defined as MK = (*mandocca* – mean planet) where *mandocca* is the mean apogee.

Āryabhaṭa (b.476 CE) takes the peripheries of the sun and the moon as constants at 13.5° and 31.5° respectively and those for the five planets are variable ones. On the other hand, the *Sūrya-Siddhānta* and the tables under consideration here adopt variable peripheries for all the seven bodies. Table 2.4 lists the limits of these *paridhi*s (peripheries) according to the *Sūrya-Siddhānta*.

Table 2.4: Manda Paridhis according to the Sūrya-Siddhānta

Body	*Manda Paridhi*	
	($MK = 0°, 180°$)	($MK = 90°, 270°$)
Sun	14°	13°40'
Moon	32°	31°40'
Kuja	75°	72°
Budha	30°	28°
Guru	33°	32°
Śukra	12°	11°
Śani	49°	48°

The *manda paridhi* is maximum at the end of an even quadrant (i.e. for $MK = 0°, 180°$) and minimum at the end of an odd quadrant (i.e. for $MK = 90°, 270°$).

If the peripheries at the ends of *even* and *odd* quadrants are denoted respectively by p_e and p_o, then the variable periphery for *mandakendra* is given by

$$p = p_e - (p_e - p_o) \times |\sin(MK)|, \tag{2}$$

where $|\sin(MK)|$ means the numerical or absolute value of $\sin(MK)$.

Thus, according to the *Sūrya-Siddhānta*, the *mandaphala* is given by (1) using (2). The values of *mandaphala* of the sun as per the *Grahagaṇitapadakāni* and the *Pratigāmī Gaṇitam*, for *mandakendra* at intervals of 10°, are compared with the actual ones, obtained from (1) and (2) in Table 2.5.

Table 2.5: Mandaphala of the Sun

MK	*Mandaphala (Equation of Centre)*					
	TYGMS Kavik		*PRB Kavik*		*Modern Kavik*	
10°	23	07	23	07	23	07
20°	45	19	45	19	45	21
30°	66	03	66	03	66	03
40°	84	36	84	35	84	37
50°	100	31	100	33	100	33
60°	113	25	113	21	113	24
70°	122	47	122	47	122	50
80°	128	31	128	32	128	36
90°	130	31	130	31	130	31

In Table 2.5, we have compared the *mandaphala* values for the sun whose *manda paridhi* varies from 13°40' to 14°. We notice that the values differ by a maximum of 5 arc seconds.

According to the Indian classical texts, the greatest *mandaphala* among the seven heavenly bodies is for Kuja (Mars) whose *manda paridhi* varies from 72° to 75°. For *mandakendra* = 90°, the *manda paridhi*, $p = p_o = 72^o$ so that the corresponding *mandaphala* = $72^o/2\pi \approx 11^o27'33'' = 687'33''$. To examine how the *mandaphala* values for a planet according to the *saura-pakṣa* tables under consideration compare with one another, these are shown in Table 2.6.

We notice in Tables 2.7 and 2.8 that (i) the *Grahagaṇitapadakāni* gives the *mandaphala* of Kuja, Budha and Guru only in *kalā*s, to the nearest arc minute while the *Pratigāmī Gaṇitam* provides the same both in *kalā*s and *vikalā*s. We notice that the values almost coincide with a difference of few arc seconds.

Table 2.6: Mandaphala of Kuja

MK	*Mandaphala (Equation of Centre)*				
	TYGMS	*PRB*		*Formula*	
	Kalās	*Kalās*	*Vikalās*	*Kalās*	*Vikalās*
10°	123	123	31	123	31
20°	242	241	31	241	48
30°	352	351	31	351	32
40°	449	449	23	449	48
50°	534	533	47	533	58
60°	602	601	57	601	49
70°	651	651	15	651	36
80°	681	681	29	681	59
90°	692	692	03	692	13

Table 2.7: Mandaphala of Budha

MK	*Mandaphala (Equation of Centre)*				
	TYGMS	*PRB*		*Formula*	
	Kalās	*Kalās*	*Vikalās*	*Kalās*	*Vikalās*
10°	49	49	10	49	10
20°	96	96	41	95	45
30°	138	138	28	138	30
40°	176	176	12	176	19
50°	208	208	11	208	22
60°	234	233	53	233	57
70°	252	252	24	252	33
80°	264	263	41	263	51
90°	268	267	34	267	39

Table 2.8: Mandaphala of Guru

MK	*Mandaphala (Equation of Centre)*				
	TYGMS	*PRB*		*Formula*	
	*Kalā*s	*Kalā*s	*Vikalā*s	*Kalā*s	*Vikalā*s
10°	54	54	26	542	6
20°	107	106	36	106	40
30°	155	155	11	155	13
40°	199	198	33	198	43
50°	236	235	47	235	58
60°	266	266	0	266	0
70°	288	287	54	288	1
80°	301	301	19	301	27
90°	305	305	58	305	58

In *fig*. 2.4 the variation of the *mandaphala* with the *mandakendra* (anomaly from the apogee) is shown graphically for the five planets. The behaviour of the graphs is sinusoidal with *MP* = 0° for *MK* = 0°, 180° and reaching the maximum at *MK* = 90°.

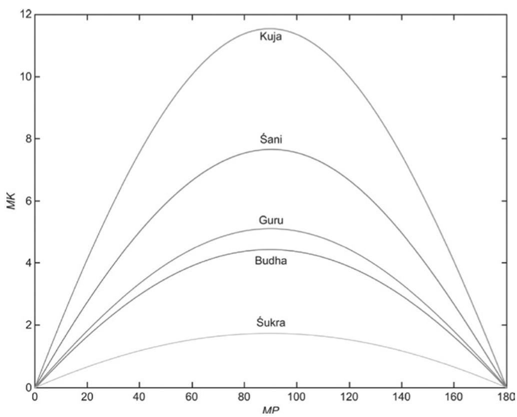


fig. 2.4: Variation of *MP* of the planets *MK*

Śīghraphalas in Pratigāmī Gaṇitam and Grahagaṇitapadakāni

As pointed out earlier in obtaining the true planets we apply two major equations which are referred to as the *manda-saṁskāra* and the *śīghra-saṁskāra.* While the former corresponds to the equation of centre, the latter to the transformation from the heliocentric to the geocentric frame of reference for the five *tārāgraha*s.

The classical procedure for *śīghraphala*s is based on the expression:

$$\mathrm{Sin}(SP) = \frac{P}{SKR}\left[R\sin(SK)\right],$$

where SP is the required *śīghraphala*s, p is the *śīghraphala*s, the periphery of the *śīghra* epicycle, R = 3438' and SKR is the *śīghrakarṇa*, the *śīghra* hypotenuse given by

$SKR^2 = (sphuṭakoṭi)^2 + (doḥphala)^2$. (4)

Example: Find the *śīghra* correction for Śani (Saturn) given the following:

Śani's *śīghrakendra* = 62°.0406 and Śani's corrected *śīghra paridhi*, p = 39°.88328.

We have

i. *Doḥphala* = $\frac{39°.88328}{360} \times 3438' \times \sin(62°.0406) = 336''.4284$. (5)

ii. *Koṭiphala* = $\frac{39°.88328}{360} \times 3438' \times \cos(62°.0406) = 178''.5765$. (6)

iii *Sphuṭakoṭi* = $3438' + 178'.5765 = 3616'.5765$. (7)

iv. *Śīghrakarṇa* = $\sqrt{(336'.4284)^2 + (3616'.5765)^2} = 3632'.1907$. (8)

v. R sin (SP) = $\frac{3438' \times 336'.4284}{3632'.1907} = 318'44166$. (9)

vi. *Śīghraphala*, SP = $\sin^{-1} = \left[\frac{318'.44166}{3438'}\right] = 5°18'53''$. (10)

The *śīghraphala* is additive or subtractive according as the *śīghrakendra* is less than or greater than 180°.

In the above example, since SK = 62°.0406 < 180°, SP > 0, i.e. SP = + 5°18'53''.

It should be noted that in the case of the *śīghra* correction also, as for the *mandaphala*, the *śīghra paridhi* (periphery) p is a variable given by

$$p = p_e - (p_e - p_o) \times \left|\sin(SK)\right| \tag{11}$$

The peripheries p, for different planets, at the ends of *even* and *odd* quadrants according to the *Sūrya-Siddhānta* are given in Table 2.9.

The *śīghra paridhi* for Kuja, Budha and Śukra is greater at the end of the *even* quadrants (*SK* = 0°, 180°) than at the *odd* quadrants (*SK* = 90°, 270°). But it is the other way for Guru and Śani.

Among the five *tārāgraha*s, Śukra (Venus) has the maximum *śīghra paridhi* and hence we choose to tabulate its value according to the different *sāriṇī*s and *padaka*s, at intervals of 15° for *SK* = 0° to 180° in Table 2.10.

In Tables 2.10-12 the *śīghrapahala* of Śukra, Kuja and Budha are compared according to the two astronomical tables, the *Pratibhāgī Gaṇitam* and the *Grahagaṇitapadakāni* with the corresponding values according to those obtained from formula based on the *Sūrya-Siddhānta*, as the tables are based on the *Sūrya-Siddhānta*.

Table 2.9: Śīghra Paridhi of Planets

Planet	*śīghra Paridhi*	
	SK = 0°, 180°	*SK* = 90°,270°
Kuja	235°	232°
Budha	133°	132°
Guru	70°	72°
Śukra	262°	260°
Śani	39°	40°

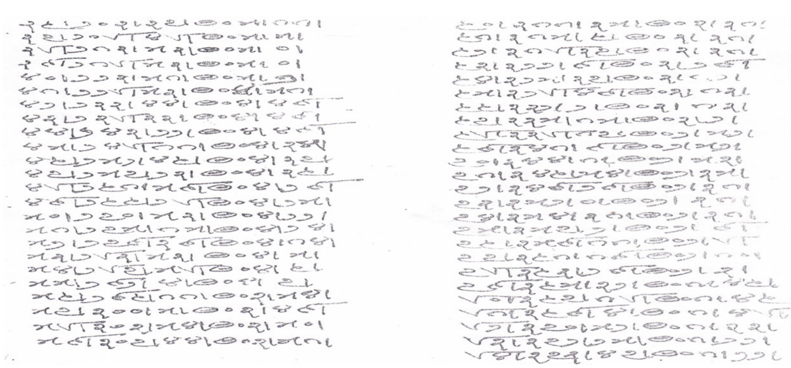


fig. 2.5: Śīg*hrapadaka* of Śani, a folio from the *Pratibhāgī Gaṇitam*

Table 2.10: Śīghrapahala of Śukra

SK	*PBR*	*TYGMS*	*Modern*
0°	0°	0°	0°
15°	6°18'17"	6°18'	6°18'16"
30°	12°32'19"	12°33'	12°33'14"
45°	18°42'21"	18°42'	18°42'13"
60°	24°43'32"	24°44'	24°41'47"
75°	30°27'32"	30°28'	30°27'01"
90°	35°51'32"	35°52'	35°50'16"
105°	40°39'06"	40°39'	40°38'19"
120°	44°27'30"	44°28'	44°26'16"
135°	46°23'05"	46°23'	46°21'23"
150°	44°16'37"	44°17'	44°14'56"
165°	32°14'13"	32°14'	32°12'36"
180°	0°	0°	0°

Table 2.11: Śīghrapahala of Kuja

SK	*PBR*	*TYGMS*	*Modern*
0°	0°	0°	0°
30°	703'44"	703'	703'44"
60°	1375'59"	1376'	1374'40"
90°	1968'53"	1969'	1967'58"
120°	2374'8"	2374'	2372'27"
150°	2191'22"	2191'	2189'57"
180°	0°	0°	0°

Table 2.12: Śīghrapahala of Budha

SK	*PBR*	*TYGMS*	*Modern*
0°	0°	0°	0°
30°	476'29"	477'	476'39"
60°	902'1"	902'	902'0"
90°	1209'10"	1209'	1208'10"
120°	1276'40"	1278'	1276'17"
150°	906'58"	907'	906'59"
180°	0°	0°	0°

In Table 2.10 the first column the *śīghrakendra*, the "anomaly of conjunction" is taken from 0° to 180° at intervals of 15°. In Tables

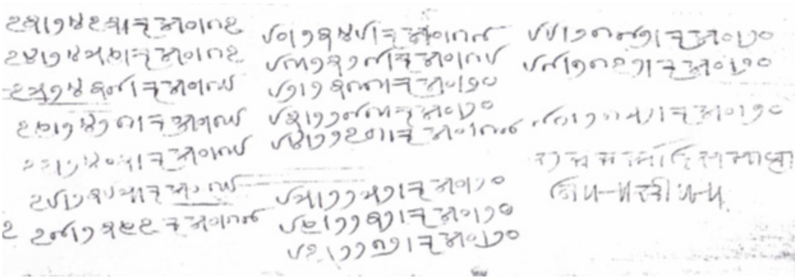


fig. 2.6: Śukra's (*karkādi*) *śīghraphala*, a folio from *TYGMS*

2.11 and 2.12 the first column the *śīghrakendra*, the "anomaly of conjunction" is taken from 0° to 180° at intervals of 30°.

The *Pratibhāgī Gaṇitam* gives the *śīghrapahala* values in *kalā* and *vikalā*s and the *Grahagaṇitapadakāni* only in *kalā*s. We notice that the three texts of *sāriṇī*s (or *padaka*s) are loyal to the basic text *Sūrya-Siddhānta* on which these are based and their *śīghrapahala* values are much closer to the formula-based last column.

Conclusion

In this paper we have studied mean motion, revolutions, sidereal periods, *mandaphala* and *śīghraphala* according to the *Pratibhāgī Gaṇitam* and the *Grahagaṇitapadakāni* manuscripts and compared their values with the modern formula.

References

Grahalāghava of Gaṇeśa Daivajña, S. Balachandra Rao and S.K. Uma, Eng. Exposition, Math. Notes, etc., *IJHS* (2006), 41.1-4, INSA, 2006.

Karaṇakutūhalam of Bhāskara II, S. Balachandra Rao and S.K. Uma, an Eng. tr. with Notes and Appendices, *IJHS* (2007), 42.1-2; (2008) 43.1 & 3.

Rao, S. Balachandra, 2000, *Ancient Indian Astronomy*: *Planetary Positions and Eclipses*, Delhi: B.R. Publishing Corp.

———, 2005, *Indian Mathematics and Astronomy*: *Some Landmarks* (rev. 3rd edn), Bangalore: Bhavan's Gandhi Centre of Science & Human Values.

Rao, S. Balachandra and Padmaja Venugopal, 2009, *Transits and Occultation's in Indian Astronomy*, Bangalore: Bhavan's Gandhi Centre of Science & Human Values.

3

An Interesting Manuscript Dealing with Algebra

Sita Sundar Ram

Abstract: The *Bījagaṇita* of Bhāskarācārya of the twelfth century forms the second part of his magnum opus *Siddhāntaśiromaṇi.* The *Sūryaprakāśa* of Sūryadāsa and the *Bījapallava* of Kṛṣṇa Daivajña are the commentaries available to us. The text *Bījagaṇita*, from the fourth chapter to the end of the text with the commentary *Sūryaprakāśa* have been taken for critical edition and translation as a project under the Indian Science National Academy. Several manuscripts have been collated to arrive at an error-free text. Since the *Bījapallava,* the other commentary is already available as an edited text; some comparison could be done for alternate readings. In this paper, the manuscript *Sūryaprakāśa* of Sūryadāsa has been analysed from different angles to highlight the contribution of Sūryadāsa.

Keywords: *Sūryaprakāśa,* Sūryadāsa, *Bījagaṇita,* commentary, manuscript.

THE BĪJAGAṆITA of Bhāskarācārya of the twelfth century forms the second part of his magnum opus *Siddhāntaśiromaṇi*. It is one of the earliest texts devoted entirely to algebra. According to Dr Pingree in his Census of the exact sciences, there are at least six commentaries on the *Bījagaṇita*. Of these, the *Sūryaprakāśa* of Sūryadāsa and the *Bījapallava* of Kṛṣṇa Daivajña are available to us.

The *Bījapallava* has been edited and published from Varanasi, Tanjore and Jammu. The first three chapters of the *Sūryaprakāśa* from the beginning to the chapter on Kuṭṭaka were taken up for doctoral thesis by Pushpakumari Jain. This has been published by MS University, Baroda. The rest of the text, from the fourth chapter to the end of the text have been taken for critical edition and translation as a project under the Indian Science National Academy. Several manuscripts have been collated to arrive at an error-free text.

The manuscripts (of *Sūryaprakāśa* of Sūryadāsa) compared are:

(क) India Office, London, 2824 (1891), ff.71.

(ख) Prajnapathasala Mandala, Wai 9777/11-2/551.

(ग) British Library, San I.O. 1533a.

(घ) British Museum, London, 447, ff.46, nineteenth century.

(ङ) British Museum, London, 448, ff.40, nineteenth century.

Problems Identified

- Legibility was very poor in three manuscripts in (क), (ख) and (घ).
- Two of the manuscripts had a number of mathematical errors – for instance, the numbers were wrongly given; the denominators were missing in the fractions (ङ) and (घ).
- Portions of the text were deranged in manuscript (ख).
- The manuscripts had to be deligently studied and compared to avoid mathematical errors.
- The *sūtra*s giving the rules and the examples were found only in manuscripts (ग) and (ङ) and missing in (क), (ख) and (घ).

Omissions

The following are instances where a manuscript omits an important but it is found in another.

1. (क) omits *भिन्नभागहारविधिनारूपशेषे ह्रियमाणेछेदांशविपर्यासोभवतिइति अयमर्थः।* (Ekavarṇa).

2. (ख) omits या ३९/५६ रू० । या ० रू १३ /६४ (Ekavarṇa).

3. अनयोः अन्तरे क्रियमाणे योगं करण्योर्महतीं इति महती करणी ९३२१/१६९ वधः १९३०५६००/२८५६१ मूलं ५१६०/१६९ द्विगुणं लघुश्च ८३२०/१६९। अनयोः रूपवत् अन्तरे कृते जातं १/१६९।। was the version in some manuscripts. The corrected version is given below. (Vargaprakṛti)

   अनयोः अन्तरे क्रियमाणे योगं करण्योर्महतीं इति महती करणी ८३२१/१६९ वधः १७३०५६००/२८५६१ मूलं ४१६०/१६९ द्विगुणं लघुश्च ८३२०/१६९। अनयोः रूपवत् अन्तरे कृते जातं १/१६९।। (Vargaprakṛti)

4. Here the necessary passage is added. Before एवं द्विघ्नकनिष्ठेनङ्गुण्ण (क)ए (घ) and (ङ) add तथा ज्येष्ठमूलं साधयितुं द्विघ्नः कनिष्ठवर्ग एव इष्टः कल्पितः। (Vargaprakṛti).

5. (ङ) omits कार्या। चत्वारः क्षेपे ययोः ते चतुःक्षेपैः द्वौ क्षेपौ ययोः ते द्वि क्षेपे। च ते च मूले च ताभ्यां रूपक्षेपार्थंए भावना। (Vargaprakṛti)

Mathematical Errors

1. न्यासः क १ ज्ये ३ क्षे २ँ।
   क १ ज्ये ३ क्षे २ँ।

   (ङ) wrongly reads क्षे २. This is a very grave mathematical error found in the particular manuscript. (Vargaprakṛti)

2. The correct reading here is मूले क १६१/५ ज्ये ५३४/५ क्षे १। (घ) reads क १५६१ ज्ये ५५३४ य (क) and (ग) read क १६१ ज्ये ५३४ leaving out the denominators. (Vargaprakṛti)

3. एवं एकद्विचतुर्मितेषु क्षेपेषु is the correct reading which was only in one manuscript. The said rule is not applicable when the additive is three as is given below. (Cakravāla)

   (क), (ख), (ग) and (ङ) read एकद्वित्रिचतुर्मितेषु

4. तथा कृते न्यासः क ८ ज्ये २७ क्षे १ is the right reading. But manuscript (ङ) reads ज्ये ८ (Cakravāla)

5. न्यासः क ३/२ ज्ये ११/२ क्षे १। is correct. (क) and (ग) read क्षे ५. With additive as 5, the correct solution is not obtained.

Emendation

1. उक्तवत् गुणाप्ती च [५ए११]।

   (क), (ख), (ग), (घ) and (ङ) read ११ ५ which is incorrect. The corrected version has been indicated within square brackets.

2. Again (क), (ख), (ग), (घ) and (ङ) read क १/२ whereas the corrected version is एवं मूले [क १/३] ज्ये २/३ क्षे १ँ। It has been indicated within square brackets.

3. अथ न्यास: प्र ५ँ क्षे २१। (क), (ख), (ग), (घ) and (ङ) wrongly add द्वितीयमूलस्य अपि भावनार्थं. There is no second root to be found in this example.

4. चतुर्णां वणिजां अश्वाः क्रमेण पंचगुणांगमंगलमिता इति। अथ चतुर्णां[1] [उष्ट्राश्च द्विमुनिश्रुतिक्षितिमिता]। तथा चतुर्णां अश्वतरावाम्यश्च अष्टद्विमुनिपावकः। तथा चतुर्णां वलीवर्दाः वृषाः मुनिमहीनेत्रेन्दुसंख्याः आसन्।

   The four traders have 5 , 3, 6 and 8 horses, 2, 7, 4 and 1 camel, 8, 2, 1 and 3 mules and 7, 1, 2 and 1 ox respectively.

   All the manuscripts have left out the number of mules. It had to be added. (Anekavarṇa)

5. भो महीपते चेदेभिर्द्रम्मैः एतदेवाप्यते विनोदार्थं has been amended as [भो सखे एभिर्द्रम्मैः एतदेवाप्यते महीपतेः विनोदार्थं].

   The lines as they appear in the MSS seem to be addressed to the king, whereas it actually means "Oh (friend) please bring for the amusement of the king, 100 pigeons and other such birds amounting to 100 for a price of 100 *dramma*s" (Anekavarṇa).

Wrong Placement

कुट्टकविधिना ... इत्यर्थः comprising of fourteen lines was wrongly placed in (ख) leading to a lot of confusion in reading the text. Comparing with other manuscripts helped in putting the entire section in its proper place.

[1] All omit [उष्ट्राश्च द्विमुनिश्रुतिक्षितिमिता].

Verse Not Found in Sūryaprakāṇa

The following example on interest rates is not found in the *Sūryaprakāśa* but in the *Bījapallava.*

एकक शत दत्त धनात् फलस्य वर्गं विशोध्य परिशिष्टम्।
पंचकशतेन दत्तं तुल्यः कालः फलं च तयोः।।

Information in Colophon

दैवज्ञज्ञानात्मज सूर्याभिधानप्रोक्ते सद्बीजभाष्ये सुजनबुधजनानंदसंदोहहेतौ ।
संयक् सूर्यप्रकाशे पटुवटुहृदयध्वान्तविध्वंसदक्षे तूर्णं पूर्णं तु तद्वद्द्विविध-
मतिभरैरेकवर्णाख्यबीजम्।

Sūryadāsa here says that he is the son of the astrologer Jñānarāja; he has written the commentary called *Sūryaprakāśa* for the text *Bījagaṇita*; and this is the chapter dealing with equations with one unknown. (Ekavarṇa)

This information about his father and the names of the text and chapter are found at the beginning and end of every chapter.

Different Reading

Since the *Bījapallava*, the other commentary is already available as an edited text, some comparison could be done for alternate readings:

1. The following verse which explains the method to solve quadratic equations is taken from the extant algebra text of Śrīdharācārya and quoted by Sūryadāsa. (Madhyama)

   चतुराहतसमै रूपैः पक्षद्वयं गुणयेत्।
   अव्यक्तवर्गरूपैर्युक्तौ पक्षौ ततो मूलम।।

   The first line being the same, the second line is quite different in the *Bījapallava* of Kṛṣṇa Daivajña. It is as follows:

   चतुरागतसमरूपैः पक्षद्वयं गुणयेत्।
   पूर्वाव्यक्तस्य कृतेः समरूपाणि क्षिपेत्तयोरेव।।

   Both explain Śrīdhara's method but the readings are different.

2. In the following example, Sūryadāsa has taken the reading *daśayuk* meaning "along with ten", against Kṛṣṇa who uses the reading *daśabhuk* meaning "after spending ten". Both, therefore, have different solutions.

पुरप्रवेशे दशदो द्विसंगुणं विधाय शेषं दशयुक् च निर्गमे।
ददौ दशैवं नगरत्रयेऽभवत् त्रिनिघ्नमाद्यं वद तत् कियत् धनम्।।

A trader paying Rs. 10 as tax on entering a town, doubled his remaining capital and paid Rs. 10 as exit tax. Thus in three towns (visited by him) his original capital tripled. Tell me what was the original capital? (Ekavarṇa)

Sūryadāsa's Solution

Let the trader's original capital be x

After giving tax in first city, the money he had = $x - 10$

After the wealth doubled, it is = $2x - 20$

After giving away 10 more, it is = $2x - 30$

After giving tax in second city, the money he had = $2x - 40$

After the wealth doubled, it is = $4x - 80$

After giving away 10 more, it is = $4x - 90$

After giving tax in third city, the money he had = $4x - 100$

After the wealth doubled, it is = $8x - 200$

After giving away 10 more, it is = $8x - 210$

Now his capital has tripled; therefore, $8x - 210 = 3x$

Solving the equation, his original capital is $x = 42$.

Kṛṣṇa's Solution

Let the trader's original capital be x

After giving tax in first city, the money he had = $x - 10$

After the wealth doubled, it is = $2x - 20$

After spending 10 and giving away 10 more, it is = $2x - 40$

After giving tax in second city, the money he had = $2x - 50$

After the wealth doubled, it is = $4x - 100$

After spending 10 and giving away 10 more, it is = $4x - 120$

After giving tax in third city, the money he had = $4x - 130$

After the wealth doubled, it is = $8x - 260$

After spending 10 and giving away 10 more, it is = $8x - 280$

Now his capital has tripled; therefore, $8x - 280 = 3x$

Solving the equation, his original capital is $x = 56$.

Mathematical Innovation

> इत्येतदर्थं अस्माभिः स्वगणिते पक्षद्वयस्य वर्गीकरणव्यतिरेकेणापि सिद्धमूलानयनप्रकारोऽभि विहितः। स यथा।
>
> The method to arrive at the square root on both sides without resorting to squaring of the terms has been explained by us in our (algebra) text. This is as follows:
>
> अव्यक्त वर्गो द्विगुणो विधेयश्च अव्यक्त एव परिकल्प्य रूपम्।
> वर्णाहतोन्यो द्विगुणस्य रूपवर्गान्वितः तत् पदमत्र मूलम्।
>
> Double the coefficient of the square of the unknown. (This is now the unknown term.) Keep the coefficient of the first degree term as the absolute number. (This is one side.) On the other side, add twice the product of the (new) coefficient of the unknown and the absolute term to the square of the (new) absolute term. Equating the two sides yields the square roots. (Madhyama)

To explain Sūryadāsa's method:

Let $ax^2 + bx + c = 0$ be the given equation.

Then $ax^2 + bx = -c$.

Then according to Sūryadāsa, on one side we take $2ax + b$; on the other we take $-4ac + b^2$. Then equate the two sides.

$$\pm(2ax + b) = \sqrt{b^2 - 4ac}$$

Example: Let the equation be $2x^2 - 9x = 18$.

According to Sūryadāsa's rule, the square root on the unknown side is $4x - 9$. On the other side, multiply the absolute number 18 by the coefficient of unknown 4. This is equal to 72. Twice 72 is 144;

adding the square of the coefficient of the first degree term of the unknown (9^2). So r.h.s. is 225 and its square root is 15.

$$4x - 9 = 15.$$

Solving, the value of the unknown x is obtained as 6.

Interesting Information

The following is an example about rice, lentils and costs, where Sūryadāsa adds some interesting information.

EXAMPLE 1

सार्धं तण्डुलमानकत्रयमहो द्रम्मेण मानाष्टकं मुद्गानां च
यदि त्रयोदशमिता एता वणिक् काकिणी।
आदायाऽर्पय तण्डुलांशयुगलं मुद्गैकभागान्वितं क्षिप्रं
क्षिप्रभुजो व्रजेम हि यतः सार्थोऽग्रतो यास्यति।।

If three and a half measures of rice can be had for 1 *dramma* and 8 measures of green gram can be had for the same amount, take these 13 *kākiṇī*s, Oh merchant! and give me quickly two parts of rice and one part of green gram, for we must make a hasty meal and depart, since the traveller (who accompanies me) has already gone ahead.

क्षिप्रं भुनक्ति इति क्षिप्रभुक् तस्य क्षिप्रभुजः। क्षिप्रं नाम मिश्रितान्न पर्याय इति कश्चित् ... सुकृती गुर्जरदेशनिवासी पुमान् कश्चित् श्री कृष्णदर्शनार्थं द्वारकाया गन्तुं प्रवृत्तः। स तु मार्गे क्षुत्क्षामस्त्वरया भोक्तुं मार्गवैषम्यभयात्समागमेन विश्लेषो मा भूत् इति व्याकुलीभूतचित्तो वणिजं वेगेन पृच्छति इत्यर्थः।

According to some, *kṣipra* is a synonym for mixed rice. … Some person living in Gujarat on his way to have *darśana* of Lord Kṛṣṇa, became hungry and thirsty and the route being unsafe does not want to get separated from his co-traveller. Hence he wants the merchant to make haste. (Ekavarṇa)

EXAMPLE 2

यदि समभुवि वेणुर्द्वित्रिपाणि प्रमाणो गणक पवनवेगात् एकदेशे स भग्नः।
भुवि नृपमितहस्तेष्वङ्ग लग्नं तदग्रं कथय कतिषु मूलादेष भग्नः करेषु।।

If a bamboo, measuring 32 cubits, and standing upon level ground, is broken at one place by the force of the wind, and the tip of the bamboo meets the ground at 16 cubits, tell me dear mathematician, at how many cubits from the root is it broken?

भो अंग गणक द्वित्रिपाणि प्रमाणो वेणुः समभुवि पवनवेगात् भग्नो दृष्टः तदग्रं यदि मूलात् भूपमित हस्तेषु लग्नं तर्हि कतिषु हस्तेषु अयं इति प्रश्नार्थः। अत्र अंग इति संबोधनं परमप्रेमास्पदद्योतनार्थम्। तथा विधेऽपि बालकादौ कृत प्रश्नायोगादतो गणक पदम्।

Dear mathematician, if a bamboo, measuring 32 cubits, and standing upon level ground, is broken at one place by the force of the wind, and the tip of the bamboo meets the ground at 16 cubits, tell me at how many cubits from the root is it broken?

This is the meaning. Here the word *aṅga* is used to denote a lot of affection. Then again probably *gaṇaka* refers to young students (of mathematics). (Madhyama)

Poetic Fancy

Sūryadāsa goes lyrical while explaining the following example. Bhāskara has given the following verse as an illustration for quadratic equations. Sūryadāsa adds his own information about Arjuna.

EXAMPLE 1

पार्थः कर्णवधाय मार्गणगणं क्रुद्धो रणे संदधे
तस्यार्धेन निवार्य तत् शरगणं मूलैः चतुर्भिर्हयान्।
शल्यं षड्भिरथेषुभिः त्रिभिरपिच्छत्रं ध्वजं कार्मुकं
चिच्छेदास्य शिरः शरेण कति ते यानर्जुनः संदधे।।

The son of Pṛthā with great anger, took some arrows to kill Karṇa in the war. With half the number, he eliminated Karṇa's arrows. With four times the square root of the total number of arrows, he struck the horses of the chariot and sent 6 arrows against (the charioteer) Śalya. With 3 arrows he struck Karṇa's umbrella, flagmast and bow. With one arrow, he cut off Karṇa's head. How many arrows had he in all?

Sūryadāsa comments:

> यदकुलावतंसविध्वंसमुनिमनोहंसपरानन्दकन्दमुकुन्दसुन्दरपदारविन्दवन्दन-शमितशंकलाकलंको धनञ्जयो एव अविकारकृत चापकर्षण योजितशरवर्षणतः स्वसेनाहृदयशल्यमिव शल्यं स कर्णं क्षणात् अवधीत् इत्यर्थः।
>
> What it means is that Dhanañjaya or Arjuna who bowed to the lotus feet of Mukunda belonging to the Yādava clan, … killed Śalya who was like a thorn in the heart of his army, and Karṇa in a moment.

In the following example given by Bhāskara, Sūryadāsa imagines the joy of the herd of monkeys.

EXAMPLE 2

> वनान्तराले प्लवगाष्टभागः संवर्गितो वल्गति जातरागः।
> फूत्कारनादप्रतिनादहृष्टा दृष्टा गिरौ द्वादश ते कियन्तः।।
>
> In a deep dense forest, a number of monkeys equal to the square of $1/8^{th}$ of their total number was chattering away merrily. The noise and echo of their shouting were enjoyed by 12 other monkeys on the hill. What was the total number of monkeys?
>
> अयमर्थः निवितरतरुमरुमलयान्दोलितमौलिशालमालतमालतालचलच्छाखा-मृगाष्टभागः परस्परानुराग कोलाहलकिलिकिला एव संतुष्टोवलातिरभसतया नृत्यति। तथा हर्षोत्कर्षशीर्षचालनमुखविकारसीत्कारकारितशितदर्शनपरस्पर-नादविनोदमोदायमानाः कपयः पर्वते च द्वादश दृष्टा इति।

Sūryadāsa adds that one-eighth of the herd of monkeys was dancing … out of love for one another, filled with joy, making a lot of noise. This noise and echo of their shouting were enjoyed by 12 other monkeys on the hill. (Madhyama)

Other Authors

Sūryadāsa mentions several authors before his time. Some of them are familiar to us.

In at least a couple of instances, Sūryadāsa quotes the *Amarakośa* of Amarasiṁha (sixth century CE).

क्रुङ् क्रौंचोऽथ बकः कंकः पुष्कराह्वस्तु सारसः।
कोकः चक्रश्चक्रवाको रथांगाह्वयनामकः।। इत्यमरोक्तेः।

The above quotation enumerates different kinds of storks and cranes.

चक्रवालं तु मण्डलमिति अमरोक्तेः।

Amara (*kosā*) says *cakravāla* means a circle.

Sūryadāsa pays homage to Brahmagupta (son of Jiṣṇu, seventh century CE) and Caturvedācārya (Pṛthūdakaswāmi, commentator of the *Brāhmasphuṭasiddhānta,* seventh century CE). These earlier authors speak of a type (of equation) called *madhyamāharaṇa* (quadratic equations).

आद्यगणकाचार्यजिष्णुजचतुर्वेदाद्या मध्यमाहरणाख्यं भेदं वदति इत्यर्थः।
(Madhyama)

Mention of Own Work

In some places Sūryadāsa mentions his own work. These are not available now:

तत् कथमिति प्रश्नार्थः। अस्योत्तरं अस्माभिः गणितरहस्ये सम्यक् निरूपितमस्ति।

How is it possible is the question. The answer has been well explained by me in my own work *Gaṇitarahasya*.

Mention of Śulbasūtras Theorem

यतो ग्रहगणिते त्रिप्रश्नोक्तप्रथमाक्षक्षेत्रछाया द्वादशाङ्गुलशंकोर्भुजकोटिरूपत्वेन तत्कृत्योर्पदं कर्ण इति प्रसिद्धिः ।

Because in the *Grahagaṇita*, in the chapter on the Three Questions, while dealing with the first latitudinal triangle, since the twelve *aṅgula* gnomon and the shadow are taking the place of the altitude and the base, it is well known that the square root of the sum of their squares is the hypotenuse.

This is the well-known result from the Śulbasūtras, now famous as the Pythagoras Theorem.

Conclusion

It is evident that the author Sūryadāsa is not only a mathematician but also a versatile poet. He has given some beautiful descriptions while commenting on some examples. These portions both in verse and prose reveal his erudition and mathematical skills.

References

Bījapallavam of Kṛṣṇa Daivajña, ed. T.V. Radhakrishna Sastri, Tanjore Saraswathi Mahal Series No. 78, Tanjore, 1958.

Sūryaprakāśa of Sūryadāsa:

(क) India Office, London, 2824 (1891), ff. 71;

(ख) Prajnapathasala Mandala, Wai 9777/11–2/551;

(ग) British Library, San I.O. 1533a;

(घ) British Museum, London, 447, ff. 46, nineteenth century; and

(ङ) British Museum, London, 448, ff. 40, nineteenth century.

4

Edition of Manuscript Gaṇitāmṛtalaharī of Rāmakṛṣṇa

V. Ramakalyani

Abstract: Editing a Sanskrit mathematics manuscript is a challenge as it requires good vocabulary of technical words. Critical edition takes into account all the available manuscripts of the same text. The critical edition of the commentary on the *Līlāvatī*, viz. the *Gaṇitāmṛtalaharī* of Rāmakṛṣṇa, is taken up as a project by the author. Some of the salient features noticed in this manuscript will be discussed in this paper.

Keywords: Critical edition, manuscript, author, commentary.

Introduction

A CRITICAL edition or textual criticism is that which restores an author's writing to its authentic form for the sake of publication. It seeks to restore, or reconstruct, the text, as far as possible, to the form in which it could have been originally made by the author. It is a criticism, or discussion, about the text itself, i.e. the verbal expression or wording of the composition.

The critical edition of the *Gaṇitāmṛtalaharī* (*GL*) of Rāmakṛṣṇa, a commentary on the *Līlāvatī* of Bhāskara II, was undertaken in 2019 as a project for National Mission for Manuscripts, since this has not been edited and published.[1] About sixty-eight commentaries on the *Līlāvatī* are listed out in catalogues. But a few of them, viz. the *Buddhivilāsinī* of Gaṇeśa Daivajña, the *Līlāvatī-vivaraṇa* of Mahīdhara and the *Kriyākramakarī* of Śaṅkara and Nārāyaṇa are published till now. Each commentary conveys Bhāskara's ideas in its unique way and hence each one of them is important. When a manuscript is edited and published, the original text will be made available to all and hence editing of manuscripts is the need of the hour to bring out the hidden knowledge in the manuscripts to light.

The Material Required for Critical Edition

The material is of two types: primary and secondary. Primary material or critical apparatus consists of all the manuscripts of the work that are available. At first, all the available manuscripts present in different libraries are to be collected, which is not an easy task. At present, photocopy or digitized copy is available which is same as the manuscript. The secondary materials are those which are supportive of the edition. Manuscripts are of two kinds: autograph and copies; autograph is that which is written in the author's own hand and copies are reproductions of the original manuscript. It is difficult to get the autographs which were written centuries before. The handwritten copies in the manuscript libraries usually consist scribal errors and hence a few manuscripts are to be compared and collated for the critical edition.

Recording the Materials

The introduction to the edition consists of a list of the entire critical apparatus which was consulted and collated, manuscripts accepted or rejected and the manuscripts which have been collated only in part, together with the reasons.

The manuscripts of the *Gaṇitāmṛtalaharī* were collected from India Office, London – 2804; Bhandarkar Oriental Research Institute, Pune – BORI.281 of viś (i) Dāhilakṣmī XXXVIII.2;

---

[1] The book is published now – *Gaṇitāmṛtalaharī of Rāmakṛṣṇa Daivajña*, ed. V. Ramakalyani, New Delhi: National Mission for Manuscripts and D.K. Printworld, 2021.

The Royal Asiatic Society of Mumbai – BBRAS.271; Rajasthan Oriental Research Institute, Jodhpur – RORI.IV.2809, XVI.2897-98, XXV.3959 and The Oriental Institute, Baroda II.12688 (inc.). The secondary material collected like ancient commentaries and anthologies are also recorded.

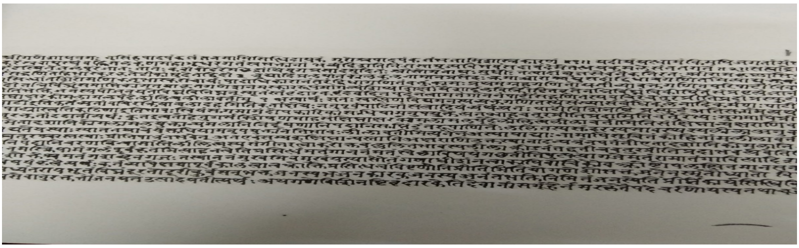
Qualifications Necessary for the Editor

Expert knowledge of the language in which the work is composed and the subject (Sanskrit and mathematics in this context) dealt with by the text is necessary for the editor. Knowledge of words employed in a secondary sense special to a particular discipline, technical terms of the subject and comprehension of the spirit of the author's entire composition are important for the editor. For example, generally *mukha* means "face". In arithmetic progression *mukha* means "the first term"; in geometry it is "the side of a figure". The synonyms of *mukha* such as *vadana* and *vaktra* are also employed in the place of *mukha*. Moreover, the editor needs to have the capacity to translate the text into English or the local language and understand the real import so that it is possible to identify the correct reading, when variant readings are seen in different manuscripts.

Deciding the Place, Family and Date of the Author

It is necessary for the author to study the introductory pages, colophon and the concluding part in the last page of the manuscript. The information about the author, his place of birth or stay, his teacher, lineage or his parents may be available in the introductory pages along with invocation. The information about the date of writing the work and also the details about the author's place and parents may be seen at the last page.

The manuscript begins as follows:



The above page is edited and given here:

श्री गणेशाय नमः ।

नृसिंहपादपंकजं नमामि सिद्धिदायकं ।
गुरोश्च पादपंकजं भजामि शास्त्रकारणम् ।।१।।

यदीयं यद्धामं किमपि जगतां संभवकरं
खवायुस्थं वह्निजलकणमहाभूधरगतम्।
स्वभक्तानां भव्यं दिशति निजसायुज्यविभवं
सदा वंदे मे तद्वयकमले संततगतम् ।।२।।

सह्याद्रेर्निकटस्थिते जलपुरे जातः कवांवोदरे
मछेद्रान्तियुगप्रसादमुदितः श्रीसोमनाथः सुधीः ।
तत्पादांबुजसेवनैकनिरतः श्रीरामकृष्णाभिधः
कुर्वे सद्गणिते हि भास्करकृते टीकां मुदे तद्विदाम् ।।३।।

त्रुट्यादिप्रलयांतकालकलनामानप्रभेदःक्रमाच्चाराश्चद्युसदां द्विधा च गणित-मित्पाद्युक्तक्रमेण सिद्धान्तशिरोमणिकर्तृभास्कराचार्यः पादाधिकारानन्तरं द्विधा गणितं वक्तुकामः प्रथमग्रहगणितोपजीव्यां व्यक्तगणितपरिपाटीं विवक्षुरादौ तन्निर्विधृसमाप्तिकामतयाकृतगणेशनमस्काररूपं मांगल्यं शिष्यशिक्षायै शार्दूलविक्रीडितवृत्तेन निर्विघ्नं चिकीर्षितं संप्रवृत्यं जानीते –

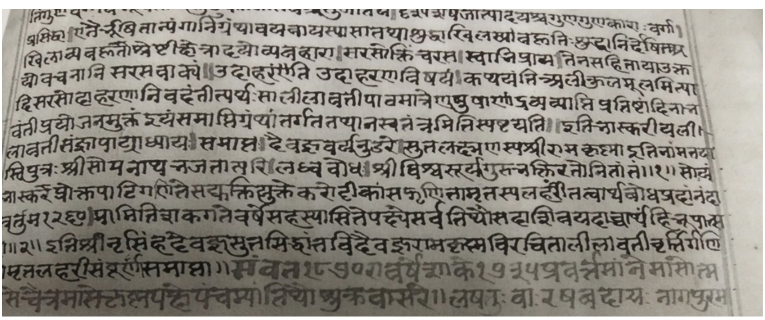
प्रीतिं भक्तिजनस्य यो जनयते विघ्नं विनिघ्नन्
स्मृतंस्तं वृंदारकवृन्दवंदितपदं नत्वा मतंगाननम्।
पाटीं स गणितस्यवचिर[2] चतुरप्रीतिप्रदां प्रस्फुटां
संक्षिप्ताक्षरकोमलामलपदैर्लालित्यलीलावतीम् ।। १।।

[2] In the edited text of *Līlāvatī*, ed. Apte 1937, it is वच्मि.

अस्यार्थः । सतः स्वरूपेण विद्यमानस्य व्यक्तगणितस्य संख्यासंबंधिकलना-
दिकर्मणः पाटीं परिपाटीं इति कर्त्तव्यतां वच्मि क्रियाबलादहमिति कर्त्राक्षेपः
ननु पूर्वपाटीनां सत्वादियं व्यर्थेत्यतः पाटीं विशिनष्टि प्रस्फुटं अतिसुगमेत्यर्थः
पूर्वपाद्यस्त्वतिकठिणा इति ।

Here, after saluting Śrī Gaṇeśa, the author Rāmakṛṣṇa salutes the lotus feet of Śrī Nṛsiṁha, who gives success and his *guru* who is the cause of all knowledge. The author Rāmakṛṣṇa introduces himself as the one who is serving the lotus feet of his *guru* Śrī Somanātha and living at Jalapura near Sahyādri Ranges (Western Ghāṭs). He also says that the author of *Siddhānta Śiromaṇi,* i.e. Bhāskara II, after Pādādhikāra, wishing to write mathematics as two parts, wrote the *Vyaktagaṇita* which is basis for the *Grahagaṇita* and he is writing the commentary for this *Vyaktagaṇita*.

The manuscript ends as follows:



The edited text is as follows:

इति भास्करीयलीलावतीसंज्ञापाटाध्यायः समाप्तः ।

दैवज्ञवर्यनृहरेः सुतलक्ष्मणस्य
श्रीरामकृष्ण इति नामतयास्ति पुत्रः ।
श्रीसोमनाथ भजतात्परिलब्धबोध
श्रीविश्वसूर्य गुरुभक्तिरतो नितांतम् ।।१।।

सोयं भास्करप्रोक्तपाटिगणिते सद्युक्तियुक्तेऽकरोट्टीकास(द्)गणितामृतस्यलहरीं
तत्वार्थबोधप्रदाम् ।

नंदाभ्रर्तुमही १२६० मिते (नदाभ्रमतु १६०९ प्र[म]मिति)[3] शकगते वर्षे
सहस्यासिते पक्षे सर्वतिथौ सदाशिवपत्यदाच्चार्थं हि भूयात्सदा ।।
(यदार्थंच्चाहिभूपात्सदा)।।२।।

इतिश्रीनृसिंहदैवज्ञसुत [दैवज्ञात्मजलक्ष्मण] सिद्धांतविदैवज्ञरामकृष्णविरचिता लीलावतीवृत्तिर्गणितामृतलहरी संपूर्णा। समाप्ता ।।

Here, it is stated that Rāmakṛṣṇa was the son of Lakṣmaṇa. He has received knowledge from Śrī Somanātha, who is Śrī Viśvasūrya. This is written in the year *nandābhrartuma*, i.e. *nanda* – 9, *abhra* – 0, *ṛtu* – 6 and *ma* (moon) – 1; which gives Śaka year 1609; but in three manuscripts it is given in numeral as Śaka 1260 which means 1338 CE. The date of the work is to be decided with other evidences.

The *Gaṇitāmṛtalaharī* does not contain the *upapatti*s like the *Buddhivilāsinī* (1545 CE) or the *Kriyākarmakarī* (1534-58 CE). The later commentaries contain elaborate explanations and proofs. This leads one to the guess that the *Gaṇitāmṛtalaharī*, which has simple explanations, could be an earlier commentary. Rāmakṛṣṇa has quoted, in his *Gaṇitāmṛtalaharī*, from the works of the mathematicians Gaṅgādhara (1434 CE), Gaṇeśa Daivajña (1545 CE), Kṛṣṇa Daivajña (1601 CE) and Munīśvara (1603 CE). From this it can be concluded that as denoted in numerals in three manuscripts Rāmakṛṣṇa does not belong to Śaka 1260 (1338 CE) and he must be later than 1603 CE. The manuscripts from Rajasthan Oriental Research Institute, give Rāmakṛṣṇa's date as Śaka 1609 in numerals also. So his date can be confirmed as 1687 CE.

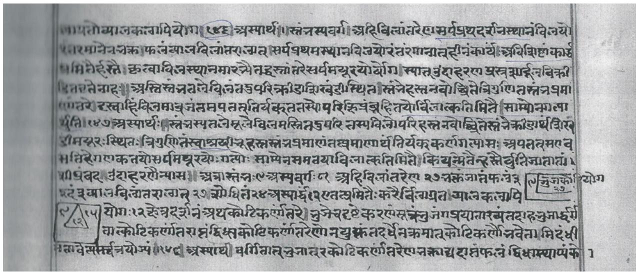
Fixing the Definitive Reading of the Text

By collating the collected manuscripts, i.e. comparing them, it is to be decided which among the variant readings is the possible correct one. The principles to be followed: obvious mistakes of the scribe can be corrected; it is possible that older copy is closer to the original; a reading that violates the rule of grammar can be rejected; internal evidence, i.e. the method in general by which the author deals with his topic and the overall manner in which he expresses himself; as this is a mathematics text, the correctness

[3] The words in the parentheses are variant readings.

of calculations can be taken into account to decide the correctness of the reading.

Let us consider an example of *bhuja-koṭi-karṇa nyāya*:



Following the *bhuja-koṭi-karṇa nyāya* (known as Pythagoras Theorem now) a rule is given to find the base and hypotenuse separately when sum of the base and hypotenuse and altitude are known. A copy of the page from one manuscript is given here. The variant readings are given in the brackets in the edited text below.

The edited text is as follows:

स्तम्भस्य वर्गोऽहिबिलान्तरेण भक्त: फलं व्यालबिलान्तरालात्।
शोध्यं तदर्धप्रामितै: करै: स्याद्बिलाग्रतो व्यालकलापियोग:।।

अस्यार्थ: स्तंभस्य वर्ग अहिबिलान्तरेण सर्पप्रथमदर्शनस्थान[4] (सर्पप्रथदर्शनस्थाने, सर्पस्थान)[5] बिलयोरंतरमानेन भक्त फलं व्यालबिलांतरालात् सर्पप्रथमस्थानबिलयोरन्तरं मानात् हीनं कार्यम् । अवशिष्टकार्ध (अविशिष्टां कार्य) प्रमितैर्हस्तै: कृत्वा बिलस्थानमारभ्यैतद्धस्तांतरे सर्पमयूरयोर्योग स्यात्। उदाहरण प्रश्नं (प्रस्त्र)शार्दूलविक्रीडितवृत्तेनाह –

अस्ति स्तंभतले बिलं तदुपरि क्रीडाशिखण्डी स्थित:
स्तंभे हस्तनवोच्छ्रिते त्रिगुणिते स्तंभप्रमाणान्तरे ।
दृष्ट्वाऽहिं बिलमाव्रजन्तमपतत्तिर्यक्स तस्योपरि
क्षिप्रं ब्रूहि तयोर्बिलात्कतिकरै: सांयेन (बिलात्कमितेसाम्योन) गत्योर्युति:।।

अस्यार्थ: – स्तंभतले मूले बिलमस्ति । तदुपरे तस्य बिलोपरि

[4] Underlined text is the preferred reading.

[5] Texts in the brackets are variant reading.

हस्तनवोच्छ्रिते स्तंभे क्रीडार्थं शिखण्डी मयूरः स्थितः । त्रिगुणिते स्वाश्रयी (स्वाश्रखी, स्वाश्रयी) भुहस्तस्तंभप्रमाणं तत्प्रमाणार्थं तिर्यक् कर्णगत्या सः अपतत् स एव मति हे गणकतयो मयूरसर्पयोः गत्योः साम्येन समतया बिलात्कतिमिते कियन्मितैः (कियन्मेतै, कियन्मितैः)न्हस्तैर्युतिजाता तां शीघ्रं वद । उदाहरणे न्यासः । अत्र स्तंभः ९ अस्य वर्गः ८१ अहिबिलान्तरेण २७ भक्तं जातं फलं ३ । इदं ३ व्यालबिलान्तरालात् २७ शोधितं २४ । अस्यार्धं १२ एतत्प्रमितैः करैर्बिलाग्रतव्याल कलापियोगः १२ ।

In the above, सर्पप्रथमदर्शनस्थान is the definite reading, as "the position of the snake that is first seen" is most suitable and the same is given in the next line of the text; अवशिष्टकार्ध is the decided reading as according to the rule, the expression finally is to be divided by 2 [see (1)] तयोर्बिलात्कतिकरैः सांयेन is the reading accepted in the texts already published and also is meaningful. स्वाश्रयी, कियन्मितैः are the suitable meaningful readings.

In the edited text above, the rule and example are given for finding the base and hypotenuse separately, when sum of base and hypotenuse and altitude are known.

The question in *Līlāvatī* 152:

> A snake's hole is at the foot of a pillar, nine cubits high; a peacock is on its top. Seeing a snake at a distance of thrice the pillar gliding towards his hole, he pounces obliquely upon him. Say quickly at how many cubits from the snake's hole they meet, both proceeding an equal distance'. – Colebrooke 1993: 97

In the *fig*. 4.1 below, the distance of meeting point C from hole B is b; the distance between hole and first position of snake, BD is $b + h$; height of pillar BA is a; then,

$$b = \frac{1}{2}\left[(h+b) - \frac{a^2}{(h+b)}\right]. \tag{1}$$

Length of the distance of the point from the foot of the pillar

$$= \frac{1}{2}\left[(27) - \frac{9^2}{(27)}\right] = 12.$$

Fig. 4.1 is drawn according to the text:

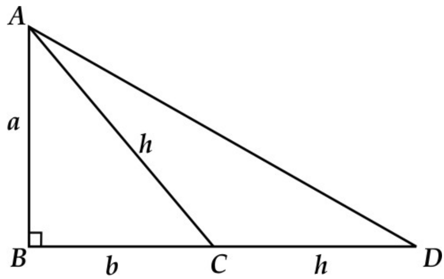


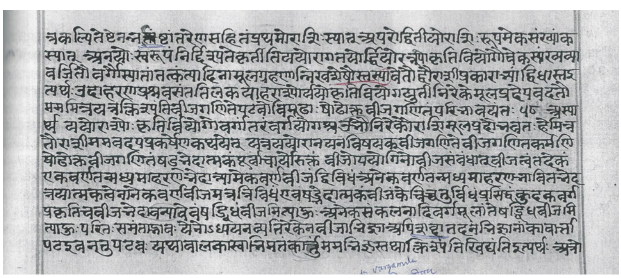
fig. 4.1: Peacock-snack problem

Some Special Features Noted

SIXFOLD ALGEBRA

Bhāskara poses a problem (*Līlāvatī* 62), “Find two quantities x and y such that $x^2 \pm y^2 - 1$ is also a square”. Then he says that those who know the six established units in algebra, in spite of being experts, find this difficult like the dull headed.

The page from the manuscript is given here:



The edited text is as follows:

राश्योर्ययोः कृतिवियोगयुती निरेके मूलप्रदे प्रवद तौ मम मित्र यत्र।
क्लिश्यन्ति बीजगणिते पटवोऽपि मूढाः षोढोक्तबीजगणितं परिभावयन्तः।।

अस्यार्थः ययो राश्योः कृतिवियोगवर्गान्तरं वर्गयोगश्च ... निरेको राशि मूलप्रदो भवतः । हे मित्र तौ राशी मम वद पप्रकर्षेण कथयेत् यत्र ययोरानयनविषयक बीजगणिते बीजगणितकर्मणि षोढोक्तबीजगणितं षड्भेदात्मकं पूर्वाचार्यैरुक्तं बीजोपयोगिनो बीजसंबन्धात् बीजत्वं तदेकं एकवर्णतन्मध्यमाहरणे भेदान्नामेकवर्णबीजं द्विविधं अनेकवर्णतन्मध्य–माहरणभावितभेदत्रयात्मकत्वेनानेकवर्णबीजमत्र त्रिविधमेवं षड्भेदात्मकं

बीजं, केचिच्चतुर्विधं प्रसिद्धं कुट्टकवर्गप्रकृति च बीजभेदा ... वेवं षड्विधं बीजमित्याहुः अनेकसंकलनादि वर्गमूलांते षड्विधबीजमित्याहुः परितः समंतात्भावः येनोऽध्ययनव्यतिरेकेन बीजाभिज्ञा अपि शक्षा तदनभिज्ञानां का वार्ता पटश्व न तु पटवः यथा बालका स्वाभिमतं का तु ममभिज्ञस्तथाक्लिश्यन्ति खिद्यन्ति इत्यर्थः।

In the second line of the *Līlāvatī* given above, षोढोक्तबीजगणितं (sixfold algebra) is explained here, which is not seen in the other known commentaries. एकवर्णबीजं द्विविधं – Equations with one variable: linear and quadratic (*madhyamāharaṇa*); अनेकवर्णबीजमत्र त्रिविधम – Equations with more than one variable: linear, quadratic and indeterminate; भावितम् – Equations with product of variables. Thus, there are six important units of algebra and this is according to the earlier *ācārya*s as he puts it "षड्भेदात्मकं पूर्वाचार्यैरुक्तं".

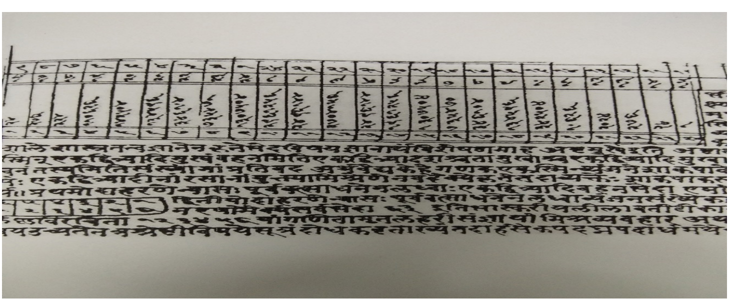
TABLE FOR COMBINATIONS

The *Līlāvatī* verse to find the total combinations of letters in a metre:

प्रस्तारे मित्र गायत्र्याः स्युः पादे व्यक्तयः कति।
एकादिगुरवश्चाऽऽशु कति कत्युच्यतां पृथक् ।।

Friend! Tell me quickly in a Gāyatrī metre how many combinations of one, two, etc. of long vowels are there in a line? How many there will be separately.

The manuscript reads as follows:



The edited version of the above page is given:

चतुश्चरणाकरसंख्याकात् चतुर्विंशत्यंकान् संस्थाप्य एकाद्यैकोत्तरा अंकाः

व्यस्ता। क्रमस्यश्च स्थापिता एतेपर पूर्वेणसंगुण्यस्तत्परस्तत्परेण चेति। संगुणक्रमस्थितां केनंविभत्य जाताः एषामेषदिभेदाक्रमेण एकः सर्वं लघुभेदाः। एतेषां योगे जाताश्चतुर्विंशत्याक्षरगायत्रीभेदाः १६७७७२१६

१	२४	१६	९	$24 = {}_{24}C_1$
२	२७६	१७	८	$24 \times 23 \div (1 \times 2) = 276 = {}_{24}C_2$
३ (२२४)	२०२४	१८	७	$24 \times 23 \times 22 \div (1 \times 2 \times 3) = 2024 = {}_{24}C_3$
४	१०६२६	१९	६	$2024 \times 21/4 = 10626 = {}_{24}C_4$
५	४२५०४	२०	५	${}_{24}C_5$
६	१३४५९६	२१	४	${}_{24}C_6$
७	३४६१०४	२२	३	${}_{24}C_7$
८	७३५४७२	२३	२	${}_{24}C_8$
९	२३०७५०४	२४	१	${}_{24}C_9$
१०	१९६१२५६	१	२४	${}_{24}C_{10}$
११	२४९६१४४	२	२३	${}_{24}C_{11}$
१२	२७०४१५६	३	२२	${}_{24}C_{12}$
१३	२४९६१४४	४	२१	${}_{24}C_{13}$
१४	१९६१२५६	५	२०	${}_{24}C_{14}$
१५	१३०७५०४	६	१९	${}_{24}C_{15}$
१६	७२५४७०	७	१८	${}_{24}C_{16}$
१७	३४६१०४	८	१७	${}_{24}C_{17}$
१८	१३४५९६	९	१६	${}_{24}C_{18}$
१९	३४२५०४	१०	१५	${}_{24}C_{19}$
२०	१०६२६	११	१४	${}_{24}C_{20}$
२१	२०२४	१२	१३	${}_{24}C_{21}$
२२	२७६	१३	१२	${}_{24}C_{22}$
२३	२४	१४	११	${}_{24}C_{23}$
२४	१	१५	१०	${}_{24}C_{24}$

Normally, Gāyatrī metre has four lines of six syllables each. Bhāskara in his *Vāsana* says "the combinations for 4 lines of 24 letters, taking the various combinations and adding them, the total number of combinations become 16,777,216 (which is = 64^4)".

Here Rāmakṛṣṇa gives a table representing combinations obtained when choosing 1, 2, …, 24 syllables, which are respectively ${}_{24}C_1, {}_{24}C_2, \ldots, {}_{24}C_{24}$.

There are more special features in the *Gaṇitāmṛtalaharī* which can be known from the text itself.

Conclusion

Critical edition of a Sanskrit text, that too a technical text like mathematics, is a challenging work. Procuring the manuscripts from different libraries is another challenge. The National Mission for Manuscripts is encouraging the scholars to edit the unpublished manuscripts. More organizations must come forward to meet this purpose so that the unknown treasure of our land can be made known to the world. Apart from mathematics, there are quite a lot of Indian astronomical manuscripts in the libraries all over the world. The youngsters should come forward to study Indian astronomy and mathematics to unravel the unstudied old texts.

References

Apte, V.G., 1937, *Līlāvatī of Bhāskarācārya*, Anandashrama Sanskrit Series, 107, Pune.

Colebrooke, H.T., 1993, *Līlāvatī of Bhāskarācārya*, New Delhi: Asian Educational Services.

Katre, S.M., 1954, *Introduction to Indian Textual Criticism,* Deccan College Postgraduate and Research Institute, Poona.

Manuscripts referred: India Office, London: 2804; Bhandarkar Oriental Research Institute, Pune – BORI.281 of viś (i) Dāhilakṣmī XXXVIII.2; The Royal Asiatic Society of Mumbai – BBRAS.271; The Oriental Institute, Baroda II.12688 (inc.).

National Catalogus Catagorum, vol. XXIV, Madras: University of Madras.

Ramachandran, T.P., 1984, *The Methodology of Research in Philosophy*, Radhakrishnan Institute for Advanced Study in Philosophy, University of Madras.

Sen, S.N., 1966, *A Bibliography on Sanskrit Works on Astronomy and Mathematics*, New Delhi: National Institute of Sciences of India.

5

Gaṇakānanda
Indian Astronomical Table

Padmaja Venugopal
S.K. Uma
K. Rupa
S. Balachandra Rao

Abstract: In this paper we present some salient features of a prominent handbook and tables belonging to the *saura-pakṣa*, based on the popular Indian astronomical treatise *Sūrya-Siddhānta* (*SS*).

Keywords: *Makarandasāriṇī* (*MKS*), *saura-pakṣa*, *Gaṇakānanda* (*GNK*), *dyugaṇa*, mean and true positions.

The *GAṆAKĀNANDA* is a popular text in Andhra and Karnataka regions. The epochal date of the text is 16 March 1447 and is based on the *Sūrya-Siddhanta*. The Telugu translation by Vella Lakshmi Nrusimha Sastrigaru of Machlipatnam is taken up. It is a handbook (*karaṇa* text) comprising of textual part and astronomical tables. The famous Andhra astronomer Sūrya, son of Bālāditya, composed his famous *karaṇa*-cum-tables, called the *Gaṇakānanda*. His more illustrious protégé Yalaya composed his exhaustive commentary *Kalpavallī* on the well-known treatise the *Sūrya-Siddhānta*.

Yalaya belonged to the Kāśyapa *gotra* and his genealogy was as follows. Kalpa Yajvā (great grandfather) – Yalaya (grandfather)

– Śrīdhara (father) – Yalaya. Yalaya quotes from his preceptors three works, viz. (i) the *Gaṇakānanda* composed in 1447 CE, (ii) the *Daivajñābharaṇa*, and (iii) the *Daivajñābhūṣana*. Yalaya's residence was a small town to the north of Addanki (latitude 15°49 N, longitude 80°01 E) called Skandasomeśvara in Andhra Pradesh. This home town of Yalaya lay towards the *āgneya* (south-east) of Śrīśaila, the famous pilgrimage centre.

Interestingly, Yalaya records some contemporary astronomical events. A few of them are the following:

i. Lunar eclipse on Saturday, Phālguna, *pūrṇima*, Śaka 1407, corresponding to 18 February 1486 CE.

ii. Solar eclipse on Friday, Phālguna *amāvāsyā*, Śaka 1389, i.e. 25 March 1468 CE.

iii. Solar eclipse on Friday, Bhādrapada *amāvāsyā*, Śaka 1407, i.e. 9 September 1485 CE, visible at his native place.

iv. Jupiter – Moon conjunction on Saturday, Āṣādha *pūrṇimā*, Śaka 1408, i.e. 17 June 1486 CE.

v. Commencement of *adhika* (intercalary) Śāvaṇa, *śukla* pratipadā, Śaka 1408, i.e. Sunday, 2 July 1486 CE.

I have verified the veracity of the above recordings by using the software prepared by me based on modern computations.

Procedure to Find Dyugaṇa for the Date 18-02-1486 according to the Gaṇakānanda Tables

In the text *Gaṇakānanda*, he considers *dyugaṇa* instead of *ahargaṇa* (heap of days from a chosen fixed epoch) for any given date, which is a very smaller unit compared to *ahargaṇa*. To find *dyugaṇa* for any given Christian day first find the Kali days from the Kali beginning and then subtract the Kali days of the epoch of *Gaṇakānanda*, 16 March 1447 CE. Now,

Kali *ahargaṇa* for the date 18-02-1486 = 1,675,402

Kali *ahargaṇa* for the epoch 16-03-1447 = 1,661,183

Therefore, *dyugaṇa* = 14,219.

To Find the mean positions of the heavenly bodies, the *Gaṇakānanda* gives the following procedures.

Multiply *dyugaṇa* (*ahargaṇa* – the number of days elapsed since the chosen fixed epoch) by 600 and divide by 16,893. The result will be in revolutions, etc. of the moon. Since the text is based on the *Sūrya-Siddhānta*, the number of revolutions of moon in a *mahāyuga* (*MY*) is 57,753,336 and the civil days in *MY* is 1,577,917,828.

i. Mean daily motion of the moon $=\frac{57753336}{1577917828}$. Hence mean position of the moon is given by

$$\frac{A\times 600}{16393}=\left(\frac{57753336}{1577917828}\right)=\left(\frac{\frac{600}{1577917828\times 600}}{57753336}\right)=\left(\frac{600}{16393.00}\right).$$

ii. *Dyugaṇa* divided by 687 gives the revolutions, etc. of Kuja (*dharaṇīsutaḥ*). Mean Kuja = $\frac{A}{687}$.

According to the *Sūrya-Siddhānta* mean Kuja =

$$\frac{2296832}{1577917828}=\frac{1}{1577917828\div 2296832}.$$

$$\text{Mean daily motion (MDM)}=\left(\frac{2296832}{1577917828}\right)$$
$$=\left(\frac{\frac{1}{1588917828}}{2296832}\right)=\left(\frac{1}{686.9974}\right).$$

Mean Kuja = $\frac{A}{687}$ (one revolution of Kuja = 687 days).

iii. Multiply *dyugaṇa* by 33 and divide by 2903 to give revolutions, etc. of the Budha *śīghrocca*.

Budha *śīghra* (*śīghrocca*) = $\left(\left(\frac{A}{2903}\right)\times 33\right)$.

$$\text{MDM of Budha}=\left(\frac{17937060}{1577917828}\right)=\left(\frac{\frac{33}{1577917828}}{17937070}\right)=\left(\frac{33}{2903.00017}\right).$$

iv. Mean Guru: Multiply *dyugaṇa* by 10 and divide by 43323 to give revolutions of Guru.

Mean Guru = $\left(\frac{A\times 10}{43323}\right)$. For $A = 1$,

$$\text{Acc. to } SS=\left(\frac{364220}{1577917828}\right)=\left(\frac{\frac{10}{1577917828\times 10}}{364220}\right)=\left(\frac{10}{43323}\right).$$

v. Mean Śukra *śīghrocca* = $\left(\frac{Dyugaṇa \times 10}{2247}\right)$ *revolutions*

For A = 1, according to *SS*, MDM

$$= \frac{7022376}{1577917828} = \left(\frac{\frac{10}{1577917828 \times 10}}{7022376}\right) = \left(\frac{10}{2246.98}\right) = \left(\frac{10}{2247}\right).$$

vi. Mean Śani =M $\left(\frac{Dyugaṇa}{10766}\right)$ revolutions = $\left(\frac{A}{10766}\right)$ revolutions

$$\text{Acc. to } SS, \text{the MDM} = \left(\frac{146568}{1577917828}\right)$$

$$= \left(\frac{\frac{1}{1577917828}}{146568}\right) = \left(\frac{1}{10766}\right).$$

vii. Moon's apogee or moon's *mandocca* = $\left(\frac{Dyugaṇa \times 10}{32321}\right)$ revolutions.

viii. Moon's node (Rāhu) = $\left(\frac{Dyugaṇa}{6794}\right)$ revolutions.

1. MDM of the sun = $\left(\frac{A \times 31}{11323}\right)$ revolutions = 0°59'81°.58''.

2. MDM of the moon = $\left(\frac{A \times 600}{16393}\right)$ = 79°34'52°84''.

3. The Telugu commentator Chella Lakshmi Narasimha Sastri has given the mean positions for the sun and the moon for his ephocal date 6 June 1856 as follows:

Mean Ravi	*53°20'38''*	*For 6 June 1856*
Epochal mean Ravi	346°52'03''	Mean positions from 12 noon
16 March 1447	66°28'35''	Motion from the epoch to the given date
Mean moon	90°30'0''	Positon for 6 June 1856
Epochal mean moon	338°46'33''	
16 March 1447	111°43'27''	Motion from the epoch to the given date

4. I have compared the mean positions of the heavenly bodies for the epoch of *Gaṇakānanda* (16 March 1447 noon) with that of *Grahalāghavam* (1520 CE), the *Sūrya-Siddhānta* and modern tropical values. It is interesting to note that the values obtained according to the various texts are comparable with the modern values (Table 5.1).

Table 5.1: Mean Epochal Positions (16 March 1447 Noon)

Body	*Mean Position Acc. to GNK for Mid-noon*	*Acc. to SS*	*Acc. to GL*	*Acc. to Modern Tropical (12^h27^m)*
Ravi	$11^R\,16°52'3''$	$11^R\,16°52'3''$	$11^R\,16°51'52''$	$0^R\,2°51'52''$
Candra	$11^R\,8°46'33''$	$11^R\,8°46'33''$	$11^R\,8°36'41''$	$11^R\,23°51'52''$
Kuja	$0^R\,12°20'46''$	$0^R\,12°20'45''$	$0^R\,13°13'29''$	$0^R\,27°27'56''$
Budha	$7^R\,1°12'44''$	$7^R\,1°12'43''$	$7^R\,12°39'51''$	$7^R\,06°51'51''$
Guru	$5^R\,8°16'20''$	$5^R\,8°16'22'$	$5^R\,6°27'36''$	$5^R\,26°27'29''$
Śukra	$11^R\,9°9'21''$	$11^R\,9°9'23''$	$11^R\,9°20'1''$	$11^R\,20°11'52''$
Śani	$3^R\,18°49'36''$	$3^R\,18°49'35''$	$3^R\,23°34''36''$	$4^R\,12°35'34''$
Candrocca	$2^R\,17°27'12''$	$2^R\,17°23'16''$	$2^R\,17°43'39''$	$3^R\,1°22'20''$
Rāhu	$0^R\,2°30'0''$	$0^R\,2°30'21''$	$0^R\,0°51'56''$	$0^R\,16°12'58''$

Note: The last column has the tropical mean longitudes tabulated. For comparison with sidereal longitudes *ayanāṁśa* (precession of equinox) has to be subtracted. Since the text is based on the *Sūrya-Siddhānta, ayanāṁśa* according to the *Sūrya-Siddhānta* is used.

$$\text{Acc. to } SS: \left((1447-522)\times\frac{54}{3600}\right) = 13°52'30''$$

$$\text{Acc. to } GL: \left(\frac{1447-522}{60}\right) = 15°25'0''$$

Procedure to Compute True Positions of the Sun, the Moon and the Planets

To find the mean positions of the sun, the moon and the planets, the method explained in the *Gaṇakānanda* is that the *dyugaṇa* is multiplied by *guṇakāra saṅkhye* (multiplier) and later divide the resulting product by the *bhāgahāra saṅkhye* (divider) continuously by multiplying the remainder at each case by 12, later by 30 and then by 60 and 60. Then the mean planet is the quotient obtained in each case after it is multiplied by 12. Table 5.2 gives the list of *guṇakāra saṅkhye* and *bhāgahāra saṅkhye* of heavenly bodies.

Table 5.2: Guṇakāra Saṅkhye and Bhāgahāra Saṅkhye of Heavenly Bodies

Bodies	*Guṇakāra Saṅkhye*	*Bhāgahāra Saṅkhye*
Sun	31	11323
Moon	600	16393
Moon's apogee (*candrocca*)	10	32321
Moon's ascending node (Rāhu)	1	6794
Mars	1	6794
Mercury	33	2913
Jupiter	10	43323
Venus	10	2247
Saturn	1	10766

MEAN AND TRUE LONGITUDE OF THE SUN FOR THE DATE 18-02-1486

Dyugaṇa for the given date 14219 is multiplied by *guṇakāra saṅkhye* 31, which gives 440789. Now dividing it by *bhāgahāra saṅkhye* 11323, it gives 38 as quotient and 10515 as remainder. Multiply the remainder 10515 by 12 and then divide it by *bhāgahāra saṅkhye*, quotient is 11 and remainder is 1627. Again multiply the remainder 1627 by 30 and divide it by *bhāgahāra saṅkhye*, quotient is 4 and remainder is 3518. Then successively multiply the remainders by 60 and find the quotients and remainders in each case, which results in quotient as 18 and remainder 7266 in one case and quotient as 38 and remainder 5686 in an other case. The first quotient 38 is the difference between the given year and the year of epoch in case of the sun is called *dhruvābda*, leaving this value consider the other quotients. Now the quotients in all cases form $11^s4^o18'38''$.

Adding epochal value to this results to mean sun

$= 11^s4^o18'38'' + 11^s16^o52'07'' = 10^s21^o10'45''$.

Therefore,

mean sun $= 10^s21^o10'45'' = 321^o10'45''$.

A new correction, called *triguṇābda* correction, is applied to mean body; according to this correction, the *dhruvābda* is multiplied by 3 and then divided that number by *triguṇābda bhāgahāra saṅkhye*

Table 5.3: Triguṇābda Bhāgahāra Saṅkhye

Bodies	*Triguṇābda Bhāgahāra Saṅkhye*
Sun	4399
Moon	2272
Mars	4297
Mercury	33239
Jupiter	20734
Venus	804
Saturn	11653
Moon's apogee	33674
Moon's ascending node	2634

successively by multiplying the remainders by 60. The *triguṇābda bhāgahāra saṅkhye* for each heavenly body is listed in Table 5.3.

According to *triguṇābda* correction, the *dhruvābda* 38 multiplied by 3 gives 114, divide this number by *triguṇābda bhāgahāra saṅkhye* of the sun 4399 successively by multiplying the remainders by 60. Which gives the quotients as 0, 1, 33 in successive cases, so 0°1'33'' is the *triguṇābda* correction for the mean sun, which has to be subtracted from the mean sun.

Mean sun – *triguṇābda phala* = 321°10'45'' – 0°1'33'' = 321°09'12''.

According to the *Gaṇakānanda* tables, the *mandocca*s of the sun and the five planets for the epoch are listed in Table 5.4.

Table 5.4: Mandocca's of Heavenly Bodies

Bodies	*Mandoccas*	*Mandocca Correction Bhāgahāra Saṅkhye*
Sun	$2^s17^o16'36''$	518
Mars	$4^s10^o02'20''$	980
Mercury	$7^s10^o27'33''$	544
Jupiter	$5^s21^o20'24''$	222
Venus	$2^s19^o51'12''$	374
Saturn	$7^s26^o37'32''$	5128

The *mandocca* correction to the given year is done to the *dhruvābda* 38 by dividing it by *bhāgahāra saṅkhye* 518 given in the above table for *mandocca* correction of the sun twice; by multiplying the remainder by 60, it gives 0 and 4 as the quotients in two cases, which is 0'4'', by adding this to the epochal *mandocca* of the sun, *mandocca* for the given year is obtained, i.e. *mandocca* of the sun for the given year = $2^s17^o16'36'' + 0'4'' = 2^s17^o16'40''$.

To find the true sun, consider *manda kendra* = *mandocca* – *triguṇābda* corrected mean sun

$$mk = 77^o16'40'' - 321^o09'12'' + 360^o$$

$$= 116^o07'28'' < 180^o$$

Therefore,

bhujā of *mk* = $116^o07'28'' - 90^o = 26^o07'28''$.

From *manda padakāntara* table of the sun, *mandaphala* for 26^o = 58'0'' for the difference 07'28'' = difference × *antara* from the table

$$= 07'28'' \times 1'30'' = 0'8''$$

Thus the *mandaphala* = $58'0'' + 0'8'' = 0^o58'8''$

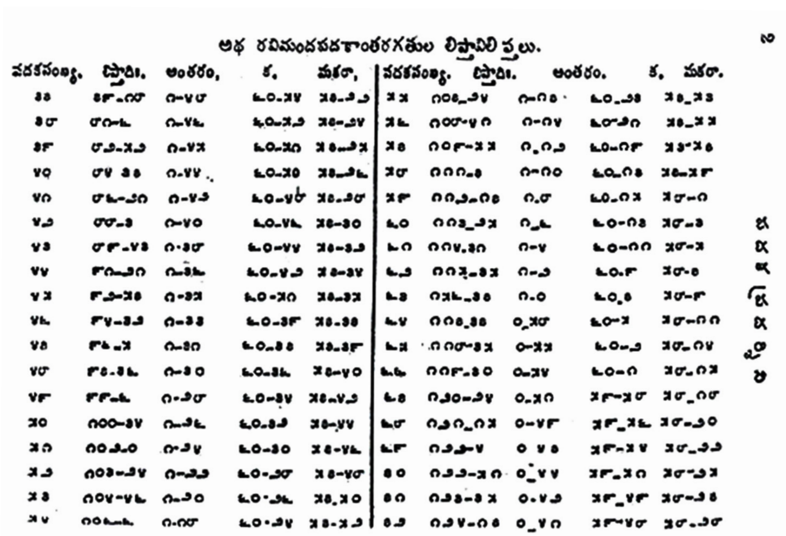


fig. 5.1: Ravi *mandapadaka* table, a folio from *Gaṇakānanda* manuscript

Since *mk* < 180°, true sun = *triguṇābda* corrected mean sun + *mandaphala*

$$= 321^\circ09'12'' + 0^\circ58'8''.$$

True longitude of the sun = 322°8'20" for the mid-noon of 18-02-1486.

MEAN AND TRUE LONGITUDE OF THE MOON FOR THE DATE 18-02-1486

To find the mean moon, *dyugaṇa* of the given date 14129 is multiplied by *guṇakāra saṅkhye* 600, which gives 8531400. Now dividing it by *bhāgahāra saṅkhye* 16393, it gives 520 as quotient and 7040 as remainder. Multiply the remainder 7040 by 12 and then divide it by *bhāgahāra saṅkhye,* quotient is 5 and remainder is 2515. Again multiply the remainder 2515 by 30 and divide it by *bhāgahāra saṅkhye,* quotient is 4 and remainder is 9878. Then successively multiply the remainders by 60 and find the quotients and remainders in each case, which results in quotient = 36 and remainder = 2532 in one case and quotient = 9 and remainder = 4382 in the other case. Neglecting the quotient obtained in the first case, the remaining quotients form = $5^s4^\circ36'9''$. To this result adding epochal value, mean moon can be obtained.

$$\text{Mean moon} = 5^s4^\circ36'9'' + 11^s08^\circ46'33''$$
$$= 4^s13^\circ22'42''.$$

After finding the mean moon the *triguṇābda* correction is applied. The *dhruvābda* 38 is multiplied by 3 gives 114; dividing this number by *triguṇābda bhāgahāra saṅkhye* of the moon 2272, successively by multiplying the remainders by 60. It gives the quotients as 0, 3, 24 in successive cases, so 0°3'24'' is the correction for the mean moon, which has to be subtracted from the mean moon.

$$\textit{triguṇābda}\text{ corrected mean moon} = 4^s13^\circ22'42'' - 0^\circ3'24''$$
$$= 4^s13^\circ19'18''.$$

Similarly, the moon's apogee (*candrocca*) is obtained for the *dyugaṇa* 206962 by using *guṇakāra saṅkhye* 10 and *bhāgahāra saṅkhye* 32321. It results to $0^s11^\circ59'05''$ adding the epoch value $2^s13^\circ27'12''$, mean *candrocca* can be obtained as $7^s7^\circ12'14''$. For this *triguṇābda* correction is applied, which results 0°0'4''.

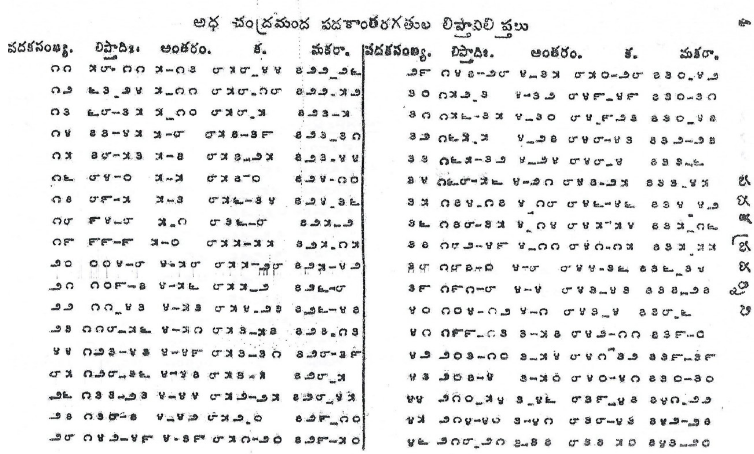


fig 5.2: *Mandapadaka* of *candra*, a folio from the *Gaṇakānanda* manuscript

triguṇābda corrected mean *candrocca* = $2^s25^o26'17'' - 0^o0'04''$

$= 7^s7^o12'10''$

To find the true moon, consider *manda kendra* = *candrocca triguṇābda* corrected mean moon

$$mk = 217^o12'10'' - 133^o19'18$$

$$= 83^o52'52'' < 180^o.$$

Therefore

$$bhuj\bar{a} \text{ of } mk = 83^o52'52''.$$

From *manda padakāntara* table of the moon, *mandaphala* for 83^o = 300'32''

And for the difference 52'52''

= difference × *antara* from the table

$= 52'52'' \times 0'39'' = 0'\,35''$

Thus, the *mandaphala*

$= 300'32'' + 0'35''= 5^o32'35''.$

Since $mk < 180^o$,

True moon = *triguṇābda* corrected mean moon + *mandaphala*

$= 133^o19'18'' + 5^o32'35''$

Therefore,

True longitude of the moon = 138°51'53'' for the midnoon of 18-02-1486.

MEAN AND TRUE LONGITUDE OF THE MARS FOR THE DATE 18-02-1486

To find the mean Mars, *dyugaṇa* of the given date 14129 is multiplied by *guṇakāra saṅkhye* 1 and dividing it by *bhāgahāra saṅkhye* 687 and successively multiplying the remainders by 12, 30, 60 and 60 as done in case of the sun and the moon. The quotients obtained are 8, 11, 0 and 16, which form as $8^s11°0'16''$.

Mean Mars = 251°0'16'' + 12°20'46'' = 263°21'02''.

Śīghrocca = 321°10'47''

After finding the mean Mars, the *triguṇābda* correction is applied. The *dhruvābda* 38 multiplied by 3 gives 114; dividing this number by *triguṇābda bhāgahāra saṅkhye* of the Mars 4297, successively by multiplying the remainders by 60. It gives the quotients as 0, 0, 1 in successive cases, so 0°0'1'' is the *triguṇābda* correction for the mean Mars, which has to be subtracted from the mean Mars. Since this value is very small, this correction is negligible.

Triguṇābda corrected mean Mars = 263°21'02''.

As the author of the *Gaṇakānanda* is the follower of the text *Sūrya-Siddhānta* (belongs to *saura-pakṣa* school), he also adopts same procedure to compute true position of the planets. The four corrections are applied to mean planets are same as that in the *Sūrya-Siddhānta* and in the following steps.

First correction (half-śīghra correction):

śīghrakendra (sk_1) = *śīghrocca* – mean planet.

Note: The mean sun is considered as *śīghrocca* for the superior planets, whereas, for the interior planets it is vice versa (it means that mean sun is considered as mean planet and mean planet is considered as *śīghrocca*). For Mars,

sk_1 = 321°10'47'' – 263°21'2'' = 57°49''45'' < 180°

fig. 5.3: *Śīghrapadaka* of Kuja, a folio from the *Gaṇakānanda* manuscript

From the above *Gaṇakānanda* tables, for *sk* = 57°, the *śīghraphala* (SE_1) = 1310'5'' and for the remaining *sk* = 49'45'' the difference in the *śīghraphala* table is considered and it is to be multiplied, i.e. 49'45'' × 21'32'' = 17'51''.

Śīghraphala (SE_1) = 1310'5'' + 17'51'' = 1327'5'' = 22°8'1''.

$$\text{Thus, first corrected Mars} = \text{mean planet} + \frac{1}{2}(SE_1)$$

$$= 263°21'02'' + \frac{1}{2}(22°8'1'')$$

$$P_1 = 274°25'2''.$$

Second Correction (Half-manda Correction)

Mandocca of Mars for the given year = 4ˢ10°02'20'' – 0'3'' = 4ˢ10°02'17'' (calculated by using the Table 5.3).

$$\textit{Mandakendra}\ (mk_1) = \textit{mandocca} - \text{first corrected Mars}\ (P_1)$$

$$= 130°02'17'' - 274°25'2'' = 215°37'15''.$$

From the *Gaṇakānanda* tables, for *mk* = 215°, the *mandaphala* (ME_1) = 402' 13'' and the remaining *mk* = 37'15'' is multiplied by the difference in the *mandaphala* table, i.e.

37'15'' × 6'46'' = 4'12''

Mandaphala (ME_1) = – 7°7'57''.

$$\text{Second corrected Mars} = P_1 + \frac{1}{2}(ME_1)$$

$$P_2 = 270°51'4''.$$

Third Correction (Full-manda Correction)

Mandakendra (mk_2) = *mandocca* – second corrected Mars (P_2)

$$= 130°02'17'' - 270°51'4'' = 219°11'13''.$$

From the *Gaṇakānanda* tables, for $mk = 219°$, the *mandaphala* (ME_2) $= -\ 7°45'24''$ by proceeding as above.

Thus, third corrected Mars = mean planet + ME_2

$$P_3 = 255°35'38''.$$

Fourth Correction (Full-śīghra Correction)

Śīghrakendra (sk_2) = *śīghrocca* – P_3

$$= 321°10'47'' - 255°35'38'' = 65°35'9'' < 180°.$$

From the *Gaṇakānanda* tables, for $sk = 65°$, the *śīghraphala* (SE_2) = 1480'40'' and the remaining sk = 35'09'' is multiplied by the difference in the *śīghraphala* table, i.e. 35'09'' × 20'56'' = 12'16''.

Śīghraphala (SE_2) = 1480'40'' + 12'16'' = 1492'56'' = 24°52'56''.

Thus, fourth corrected Mars

$$= P_3 + SE_2 = 255°35'38'' + 24°52'56'': P_4 = 280°28'35''.$$

Therefore, the true longitude of Mars = 280°28'35''.

Yalaya's example of Lunar Eclipse on 18-02-1486 is compared with modern values in the Table 5.5:

Table 5.5: Yalaya's Example of Lunar Eclipse (18 Feb. 1486)

	IST	*Modern*
Beginning of eclipse	$20^h\ 23^m$	$20^h\ 31^m$
Beginning of totality	$21^h\ 34^m$	$21^h\ 42^m$
Middle of eclipse	$22^h\ 10^m$	$22^h\ 18^m$
End of totality	$22^h\ 46^m$	$22^h\ 54^m$
End of eclipse	$23^h\ 57^m$	$24^h\ 5^m$

Table 5.6: Sun's Sidereal True Longitude for 3 April 2012

Text	*Mean Sun*	*Equation of centre*	*True sidereal Sun*
MKS	347°19'7''	2°10'32''	349°29'39''
GNK	348°2'49''	2°10'32''	350°13'20''
Modern	347°44'30''	1°54'16''	349°38'46''

This particular date is chosen since around that date every year the sun's equation of centre (*mandaphala*) is maximum. In Table 5.6, we observe that the mean longitude and the equation of centre are close in their values as per the *Makarandasāriṇī* to the modern ones. But the sun's true longitude differs from the modern value by about 9'7'' and the equation of centre (*mandaphala*) by 16'16''. These differences are mainly because in modern computations, gravitational periodic terms are considered. In the classical Indian texts, even as in European tradition before Kepler, epicyclic theory was adopted. The results obviously vary a bit compared to those of Kepler's heliocentric elliptical theory. The equation of centre (*mandaphala*) in *siddhānta*s is governed by the radii of the epicycles.

Sun's Declination (*Krānti*)

In the computations of solar eclipses and transits we need to use the declination (*krānti*) of the sun. In Table 5.7, we compare the values of the sun's declination (δ) for two days when the sun's rays fall directly on the Śivaliṅgam at the famous Ganigādhareśvara Temple in Bengaluru (see Shylaja 2008). From Table 5.7, we notice that on two days of the year 2012, viz. 14 January and 28 November, the declination of the sun has the values 21°2'29.13'' south and 21°8'51.91'' south respectively according to the *Makarandasāriṇī* and the corresponding values according to the *Gaṇakānanda* are 21°10'10.64'' south and 21°16'36.97'' south. It should be noted that the declination is calculated according to these texts for the same

Table 5.7: Sun's Declination (δ) at 17^h15^m (IST)

Text	*14 January 2012*	*28 November 2012*
MKS	21°2'29.13'' S	21°8'51.91'' S
GNK	21°10'10.64'' S	21°16'36.97'' S
Modern	21°11' S	21°17' S

time. The difference in arcminutes for the two dates according to a particular text indicates that the corresponding azimuths and the altitudes of the sun slightly differ. The difference in the values of δ according to the two classical texts as compared to the modern values is due to the fact that the Indian classical texts took the obliquity of the ecliptic as 24° while the modern known value is around 23°26'. It is significant to note that the values of the *Gaṇakānanda* are closer to the modern ones.

Transits and Occultations

The procedure for transits and occultations are similar to that of solar eclipse. The participating bodies in the case of transits will be the sun and the planets (Mercury or Venus) and for occultation moon and the planet or the star will be under consideration. The transits of Mercury and Venus occur when either of them is in conjunction with sun as observed from earth, subject to the prescribed limits. The transit of Venus is a less frequent phenomenon as compared to that of Mercury. For example, after the transit of Venus in June 2004 the next occurrence was on 6 June 2012. After that, the subsequent Venus transit will be about 105.5 years later, i.e. in December 2117.

While detailed working of planetary conjunctions is discussed in all traditional Indian astronomical texts under the chapter "Grahayuti", it has to be noted that the transits of Mercury and Venus are not explicitly mentioned. This is mainly because when either of these inferior planets is close to sun it is said to be "combust" (*asta*) and hence not visible to the naked eye. Transit (of Mercury or Venus) is called *saṅkramaṇa* (of the concerned planet) or *gadhāsta*. In a transit of Mercury or Venus the concerned tiny planet passes across the bright and wide disc of the sun as a small black dot.

Conclusion

In the preceding sections we have introduced some features of the astronomical tables belonging to the *saura-pakṣa*. Examples given by Yalaya on the lunar and solar eclipses are listed. Computing the mean positions of heavenly bodies using the procedures discussed in the *Gaṇakānanda* are explained. The mean epochal positions according to the *Gaṇakānanda* are compared with the *Sūrya-Siddhānta*, the *Grahalāghavam* and tropical mean longitudes.

References

Jhā, Paṇḍit Laṣaṇlāl, 1998, *Makaranda Prakāśaḥ*, Varanasi: Chowkhamba Surabharati Prakashan.

Rao, S. Balachandra and Padmaja Venugopal, 2008, *Eclipses in Indian Astronomy*, Bengaluru: Bharatiya Vidya Bhavan.

———, 2009, *Transits and Occultations in Indian Astronomy*, Bengaluru: Bharatiya Vidya Bhavan.

Siddhānta Gaṇakānanda-bodhinī, Chella Lakshṃi Nrisimha Sastriguru Madanapalli: Ellavāri Press, repr. 2006.

Sūryasiddhānta, tr. Rev. E. Burgess, ed. Phanindralal Gangooly with Intr. by P.C. Sengupta, Delhi: Motilal Banarsidass, 1989.

Vyasanakere, P. Jayanth, K. Sudheesh and B.S. Shylaja, 2008, "Astronomical Significance of Gavi Gangadhareshwara Temple in Bangalore", *Current Science* (December) 95(11): 1632-36.

6

Karaṇa Kutūhala Sāriṇī

Its Importance and Analysis

M. Shailaja
V. Vanaja
S. Balachandra Rao

Abstract: The tables of *Karaṇa Kutūhala Sāriṇī* are based on the *Karaṇa Kutūhala* of Bhāskara II (twelfth century). These tables are based on *brāhma-pakṣa*, though the author and period of construction of tables are not known but the manuscripts are available in libraries of oriental research institutes.

There are at least five extant manuscripts of the tables of the *Karaṇa Kutūhala Sāriṇī* with some expository details in table headings and marginal notes. For this paper we have used the manuscript of the *Karaṇa Kutūhala Sāriṇī* from BORI, Pune 501/1895–1902.

The importance of the *Karaṇa Kutūhala Sāriṇī* tables lies in that the compilers of annual astronomical almanacs (*pañcāṅga*s) of *brāhma-pakṣa* use these tables.

In this paper, the mathematical model for the construction of tables are obtained with rationales. An example is worked out to compare the results with modern ephemerical values.

Keywords: *Ahargaṇa, mandakendra, mandaparidhi, mandaphala, śīghraphala.*

Introduction

The determination of mean and true positions of the sun, the moon and the planets, computation of solar declination (*krānti*), lunar latitude (*śara*), the three problems relating to time, direction and place, risings and settings and conjunctions of the planets are the important parts of classical Indian astronomical texts.

In the *Karaṇa Kutūhala* of Bhāskara II, the above all topics are dealt with handy and simplified procedures and useful values for *kṣepaka*, *parākhya*, maximum *mandaphala*, and maximum *śīghraphala*, and the denominators to compute *mandaphala* are listed. The computation of *ahargaṇa* is also reduced by taking a contemporary date as the epoch instead of considering the beginning of *mahāyuga* as the epochal point as in his *Siddhānta-Śiromaṇi*, thereby decreasing the tedious computations into a simple way.

The tables of the *Karaṇa Kutūhala Sāriṇī* are based on the astronomical handbook *Karaṇa Kutūhala*. These tables are based on *brāhma-pakṣa* – school of astronomy adhered to by Bhāskara II, which follows the parameters of the *Brahmasphuṭasiddhānta* of Brahmagupta (628 CE). The *Karaṇa Kutūhala Sāriṇī* consists of:

i. Mean motion tables of the sun, the moon, moon's *mandocca* (apogee), moon's *pāta* and that of five planets (Mars, Mercury, Jupiter, Venus and Saturn) in days (D), months (M), years (Y) and 20-year periods (20YP).

ii. *Mandaphala* tables or tables of the equation of the centre of the sun and the moon for *manda* anomaly from 0° to 90°.

iii. Table of solar declination and of lunar latitude for the arguments from 0° to 90°.

iv. *Mandaphala* tables of planets (tables of the equation of the centre for the planets for *manda* anomaly from 0° to 90°).

v. *Śīghraphala* tables of the planets (tables of the equation of the conjunction for planets for *śīghra* anomaly from 0° to 180°).

In this paper, mainly the analysis is focused on the differences that have been introduced in the tables from the text and we have analysed how these tables of the *Karaṇa Kutūhala Sāriṇī* are useful to almanac makers to compute day-to-day calculations of motions, positions, phenomena, etc. so that it can be compiled for the entire year. Thus, the study reveals the relation between the text and the astronomical tables more precisely.

The Text: Karaṇa Kutūhala

The mean positions of the heavenly bodies are obtained by finding the number of days elapsed (*ahargaṇa A*) from the epochal date, i.e. from the mean sunrise at Ujjain on 24 February 1183 CE till the given date. Then by using the formulae and adding the epochal mean positions called *kṣepaka* to them (listed in Table 6.1), the mean positions of the heavenly bodies for the given date can be computed.

In chapter 2, the "Spaṣṭādhikāra" of the *Karaṇa Kutūhala,* Bhāskara explains the method of finding the true positions of the sun and the moon by applying the *manda saṁskāra* and to those of five planets (Mars, Mercury, Jupiter, Venus and Saturn) by applying

Table 6.1: Epochal Mean Positions (Kṣepaka) and Formulae to Find Mean Longitude

Heavenly Bodies	*Kṣepaka (K)*	*Mean Longitudes of the Body*
Sun	$10^{R}29°13'$	$\left(A-\frac{13A}{903}\right)^{\circ}+K$
Moon	$10^{R}29°05'50''$	$(14A)^{\circ}-\left(\frac{14A}{17}\right)^{\circ}-\left(\frac{A}{8600}\right)^{\circ}+K$
Mandocca of moon	$4^{R}15°12'59''$	$\left(\frac{A}{9}\right)^{\circ}+\left(\frac{A}{4012}\right)^{\circ}+K$
Moon's *pāta*	$9^{R}17°25'09''$	$\left(\frac{A}{19}\right)^{\circ}+\left(\frac{A}{2700}\right)^{\circ}+K$
Mars	$7^{R}21°14'21''$	$\left(\frac{11A}{21}\right)^{\circ}+\left(\frac{A}{52444}\right)^{\circ}+K$
Mercury's *śīghrocca*	$2^{R}21°14'30''$	$(4A)^{\circ}+\left(\frac{4A}{43}\right)^{\circ}-\left(\frac{A}{1421}\right)^{\circ}+K$
Jupiter	$2^{R}04°00'51''$	$\left(\frac{A}{12}\right)^{\circ}-\left(\frac{A}{4227}\right)^{\circ}+K$
Venus's *śīghrocca*	$8^{R}18°05'55''$	$\left(\frac{16A}{7451}\right)^{\circ}+\left(\frac{16A}{10}\right)^{\circ}+K$
Saturn	$4^{R}03°43'17''$	$\left(\frac{A}{30}\right)^{\circ}+\left(\frac{A}{9367}\right)^{\circ}+K$

two corrections called the *manda* and the *śīghra saṁskāra*s. For this purpose, the text has provided tables of *mandocca*s (apogees), *parākhya*s, maximum *mandaphala* and *śīghraphala*s. Since the model of epicycle is adopted for true positions of planets, the *manda* peripheries used are as given in the *Siddhānta-Śiromaṇi* and they are fixed. The *mandocca* of the sun is 78° and those of five planets (Mars, Mercury, Jupiter, Venus and Saturn) are respectively 128°30', 225°, 172°30', 81° and 261°.

The *mandakendra* (anomaly of equation of the centre) is the difference between *mandocca* and the mean planet.

$$Mandakendra\ (mk) = Mandocca - \text{Mean Planet.}$$

The *mandaparidhi*s of all heavenly bodies, maximum *mandaphala*s in each case and denominators to compute *mandaphala* are listed in Table 6.2.

The *mandaphala* of heavenly body is calculated by using

$$Mandaphala(MP) = \left(\frac{jy\bar{a}\left(bhuj\bar{a}\,(mk)\right)\times 10}{D}\right) = \left(\frac{R\,\text{sine}\left(bhuj\bar{a}\,(mk)\right)\times 10}{D}\right),$$

where D is the denominator of the respective planets and mk is the *mandakendra.*

For the sun and the moon, the only correction applied is equation of centre (*mandaphala*).

Table 6.2: Mandaparidhi, Maximum Mandaphalas and Denominators

Heavenly bodies	*Mandaparidhis*	*Maximum Mandaphala*	*Denominator*
Sun	13°40'	2°10'30''	550
Moon	31°36'	5°01'45''	238
Mars	70°	11°08'27''	107
Mercury	38°	6°02'52''	198
Jupiter	33°	5°15'07''	228
Venus	11°	1°45'02''	784
Saturn	50°	7°57'27''	157

Therefore,

$$\text{True sun} = \text{Mean Sun} + \textit{Mandaphala}$$

and

$$\text{True moon} = \text{Mean moon} + \textit{mandaphala.}$$

In the *Karaṇa Kutūhala,* the radius R of the deferent circle is 120° instead of the usual 360°. Corresponding to the value of radius as 120°, the *parākhya*'s of each planet is given as 81°, 44°, 23°, 87° and 13° respectively for five planets, which will be used in finding *śīghraphala* of the planet. If the radius R is taken as 360°, then *paridhi* = 3 × *parākhya.* Thus, the periphery of *śīghra* epicycle is 243°, 132°, 69°, 261° and 39° respectively for five planets.

In the case of superior planets the Mars, Jupiter and Saturn, the mean sun is considered as *śīghrocca,* for inferior planets the Mercury and Venus some special point is considered as their *śīghrocca*s and the mean sun is treated as the mean planet.

$$\textit{Śīghrakendra} = \textit{Śīghrocca} - \text{Mean Planet.}$$

In both the cases of *mandakendra* and *śīghrakendra,* generally if $0° < kendra < 180°$, then the *phala* is positive and if $180° < kendra < 360°$ the *phala* is negative.

i. *Bhujā* = *kendra,* if $kendra < 90°$.

ii. *Bhujā* = 180° – *kendra,* if $90° < kendra < 180°$.

iii. *Bhujā* = *kendra* – 180°, if $180° < kendra < 270°$.

iv. *Bhujā* = 360° – *kendra*, if $270° < kendra < 360°$.

According to the *Karaṇa Kutūhala*, the *śīghraphala* of planets is found by using the formula

$$\textit{śīghraphala} = \sin^{-1}\left[\frac{\textit{parākhya} \times \textit{bhujājyā}}{\textit{śīghrakarṇa}}\right],$$

where, *śīghrakarṇa* (SK) is given by

$$SK = \sqrt{(\textit{parākhya})^2 + 2 \times (\textit{parākhya}) \times \textit{kotijā} + (120)^2}$$

and

$$\textit{Bhujājyā} = R \sin(\textit{bhujā}),\ \textit{koṭijyā} = R \cos(\textit{bhujā}).$$

Table 6.3: Procedure to Find True Position of Planets according to Karaṇa Kutūhala

For All Planets (Except Mars)	*For Mars*
$P_1 = MP + ME_1$	$P_1 = MP + \frac{ME_1}{2}$
$P_2 = P_1 + SE_1$	$P_2 = P_1 + \frac{SE_1}{2}$
$P_3 = MP + ME_2$	$P_3 = MP + ME_2$
$P_4 = P_3 + SE_2$	$P_4 = P_3 + SE_2$

Where *ME* is the correction corresponding to the *manda* equation and *SE* corresponds to the *śīghra* equation.

To find the true positions of five planets the *manda* and *śīghra* corrections are applied successively one after the other as listed in Table 6.3.

The Tables: Karaṇa Kutūhala Sāriṇī

The *Karaṇa Kutūhala Sāriṇī* tables are the derived values of planetary mean motions with corrections for computing true motions for a given terrestrial location based on the first two chapters of the *Karaṇa Kutūhala*.

In the *Karaṇa Kutūhala Sāriṇī*, the mean motion tables are given for 1 to 30 days, then for 1 to 12 months, then for 1 to 20 years and later the table is extended to 1 to 30 periods of 20 years each (it means that the mean motion is provided for 600 years). This method of giving the motion for the period of 20-year periods is unique. The epochal values according to the *Karaṇa Kutūhala Sāriṇī* are same as that of the text *Karaṇa Kutūhala* and the date is 24 February 1183 CE. The mean daily motions are given up to fourth sub-seconds, thereby considering the fraction of motion also to the computation of positions of heavenly bodies. The mean daily motions are listed in Table 6.4.

Maximum equation of the centre (*mandaphala*) of bodies, given in the *Karaṇa Kutūhala Sāriṇī*, is slightly different as that of the *Karaṇa Kutūhala* and it attains its maximum at *manda* anomaly = 90° for all planets except for Mercury. For Mercury, the *mandaphala* is maximum for *manda* anomaly = 88°, whereas in the main text it

Table 6.4: Mean Daily Motions according to the Karaṇa Kutūhala Sāriṇī

Heavenly Bodies	*Mean Daily Motion*
Sun	$0^\circ 59' 8'' 10''' 12^{iv} 40^{v}$
Moon	$13^\circ 10' 34'' 52''' 31^{iv} 50^{v}$
Mars	$0^\circ 31' 26'' 28''' 09^{iv} 50^{v}$
Mercury's *śīghrocca*	$4^\circ 05' 32'' 21''' 01^{iv} 0^{v}$
Jupiter	$0^\circ 04' 59'' 08''' 54^{iv}\ 0^{v}$
Venus's *śīghrocca*	$1^\circ 36' 7'' 43''' 49^{iv} 50^{v}$
Saturn	$0^\circ 02' 00''\ 23''' 03^{iv} 30^{v}$
Lunar apogee	$0^\circ 06' 40'' 53''' 50^{iv} 10^{v}$
Lunar node	$-\ 0^\circ 03' 10'' 48''' 25^{iv} 30^{v}$

attains maximum at 90° for all planets. The maximum equation of the centre (*mandaphala*) for the bodies is listed in Table 6.5.

Even maximum equation of the conjunction (*śīghraphala*) of planets also differs from those of the *Karaṇa Kutūhala*. From the tables of the *Karaṇa Kutūhala Sāriṇī*, the *śīghra* anomaly at which the *śīghraphala* attains its maximum value can be easily noted. From the tables of *śīghraphala* (the equation of the conjunction) of the five planets in the *Karaṇa Kutūhala Sāriṇī*, the values of maximum equation of the conjunction is listed in Table 6.6.

Table 6.5: Maximum Equation of the Centre (Mandaphala) of the Bodies

Heavenly Bodies	*Maximum Mandaphala according to KKS*	*Maximum Mandaphala according to KK is at 90°*
Sun	2°10'54''at 90°	2°10'30''
Moon	5°02'31''at 90°	5°01'45''
Mars	11°12'53''at 90°	11°08'30''
Mercury	6°25'25'' at 88°	6°02'52''
Jupiter	5°15'47'' at 90°	5°15'30''
Venus	1°31'50'' at 90°	1°45'02''
Saturn	7°38'35'' at 90°	7°57'27''

Table 6.6: Maximum Equation of Conjunction (Śīghraphala) of the Five Planets

Planets	*Maximum Śīghraphala according to KKS*	*Maximum Śīghraphala according to KK*
Mars	41°18'16" at 130°	42°27'14"
Mercury	21°37'11" at 110°	21°30'36"
Jupiter	10°59'01" at 100°	11°03'00"
Venus	46°18'41" at 130°	46°28'08"
Saturn	06°10'24" at 100°	06°13'10"

Comparison of Karaṇa Kutūhala and Karaṇa Kutūhala Sāriṇī

The true positions of the sun, the moon and the five planets are computed according to both the *Karaṇa Kutūhala* and the *Karaṇa Kutūhala Sāriṇī* and the values are compared with the published ephemeris.

True positions of the Sun and the Moon according to the Karaṇa kutūhala

Kali ahargaṇa for 28-11-2018 = 1869985

Kali ahargaṇa for the epoch 24-02-1183 = 1564737

Difference in days = 305248, therefore the *Karaṇa Kutūhala ahargaṇa* = 305248. Here the Kali *ahargaṇa* is computed from beginning of *Kali-Yuga* (i.e. from the day between 17-18 February 3102 BCE).

Finding Mean Sun and True Sun according to Karaṇa Kutūhala

Mean Sun = $\left(1-\frac{13}{903}\right)A + K$,

where A = *Karaṇa Kutūhala ahargaṇa*, K = *Kṣepaka*

$= 10^R 29°13'$ for the sun

$= \left(\frac{890}{903}\right) \times 305248 + 10^R 29°13'$

$= 222°43'38''$

Mandakendra (*mk*) = *Mandocca* – Mean Sun

$= 78° - 222°43'38'' + 360°$

$= 215°16'22'' > 180°.$

$\therefore$ *MP* is Negative

Bhujā = *Mandakendra* – 180^0, if $180^0 < m < 270^0$

$= 35°16'22''$

Mandaphala (*MP*) = $\dfrac{R\sin(mk)\times 10}{550} = 1°15'35''$

True Sun = Mean Sun – *MP*

$= 222°43'38'' - 1°15'35'' = 221°28'03''$.

Finding Mean Moon and True Moon according to the *Karaṇa Kutūhala*

Mean moon $= \left(14 - \dfrac{14}{17} - \dfrac{1}{8600}\right)A + K,$

where $K = 10^R 29°05'50''$

$= 13.17635431 \times 305248 + 10^R 29°05'50''$

$= 104°53'51''$.

Moon's *mandocca* = $\left(\dfrac{1}{9} + \dfrac{1}{4012}\right)A + K$ where $K = 4^R 15°12'59''$

$= 0.111360363 \times 305248 + 4^R 15°12'59''$

$= 287°44'40''$.

Mandakendra (*mk*) = *Mandocca* – Mean Moon

$= 287°44'40'' - 104°53'51''$

$= 182°50'49'' > 180°$

$\therefore$ *MP* is negative.

Bhujā = *Kendra* – 180°, if 180° < *kendra* < 270°

$= 2°50'49''$.

Mandaphala (*MP*) = $\dfrac{R\sin(mk)\times 10}{238} = 0°15'02''$

True moon = Mean Moon – *MP*

$= 104°53'51'' - 0°15'02'' = 104°38'49''$.

TRUE POSITIONS OF THE SUN AND THE MOON ACCORDING TO THE KARAṆA KUTŪHALA SĀRIṆĪ

For the same date 28-11-2018, by considering the *Karaṇa Kutūhala ahargaṇa* = 305248 days and using the tables, mean and true positions of the sun, the moon and the planets are found in the following section.

After finding the *ahargaṇa* from the epochal date, divide it by 30. The integer part denotes the number of months completed and the remainder "D" in days. Then the number of months divided by 12 gives the integer number as the completed years of 360 days each and the remainder "M" in months. The number of years divided by 20 gives the number "YP"-20 year-periods and the remainder "Y" in years.

Now, 305248 days is divided successively by 30, 12, 20 to get 42YP, 7Y, 10M, 28D.

Note: In the *Karaṇa Kutūhala Sāriṇī*, the *kṣepaka* (*K*) value is already added to the values of periods of twenty years. So again adding *K* is not necessary, directly it gives the mean position. While considering the values of 20YP for more than 600 years, twice or more than twice the epochal value will be considered from the table so that *kṣepaka* (*K*) must be subtracted correspondingly once or more than once.

Finding Mean and True Sun

From the tables of the *Karaṇa Kutūhala Sāriṇī*, the mean sun is shown in *fig*. 6.1:

Motion for 30YP = $3^R09°25'41''12'''$, where $1^R = 30°$

Motion for 12YP = $5^R15°18'04''29'''$ (in both YPs the epochal value *K* is included)

Motion for 7Y = $10^R23°43'08''24'''$

Motion for 10M = $9^R25°40'51''00'''$

Motion for 28D = $0^R24°38'24''15'''$ by adding all these and subtracting

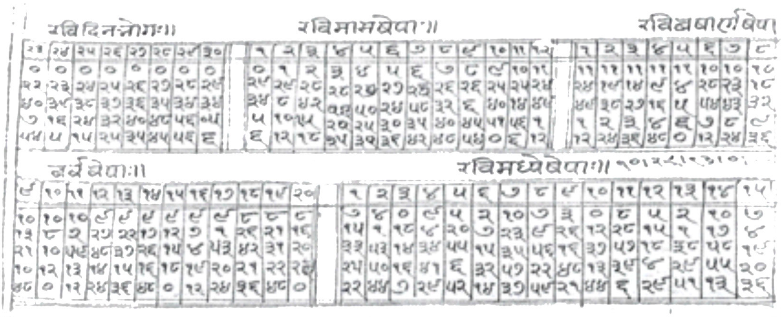


fig. 6.1: Mean motion table of the sun, a folio from the *Karaṇa Kutūhala Sāriṇī*

$K = 10^R 29°13'$ once and then by removing the cycles of 12 *rāśī*s. We get

Mean sun = $7^R 12°30'41''39''' = 222°30'41''39'''$

Mandocca of the sun = 78°

Mandakendra (*mk*) = *Mandocca* – Mean Sun

$= 78° - 222°30'41''39''' + 360°$

$= 215°29'19'' > 180°$

Bhujā = *Mandakendra* – 180°, if $180° < m < 270°$

$= 35°29'19''$

From *Ravi manda* tables (the *mandaphala* (*MP*)) is given for every degree up to 90°) (*fig*. 6.2)

Mandaphala (*MP*) = $1°14'43'' + 0°29'19'' \times 0°1'53'' = 1°15'38''$.

True sun = Mean sun + *MP*

$= 222°30'41'' - 1°15'38''$

True sun = $221°15'03'' = 7^R 11°15'03''$

Finding Mean and True Moon

Mean Moon from the tables for the date 28-11-2018 is as follows:

Motion for 30YP = $8^R 21°30'41''22'''$

Motion for 12YP = $2^R 26°06'34''19'''$

Motion for 7Y = $2^R 24°24'43''17'''$

Motion for 10M = $11^R 22°54'22''39'''$

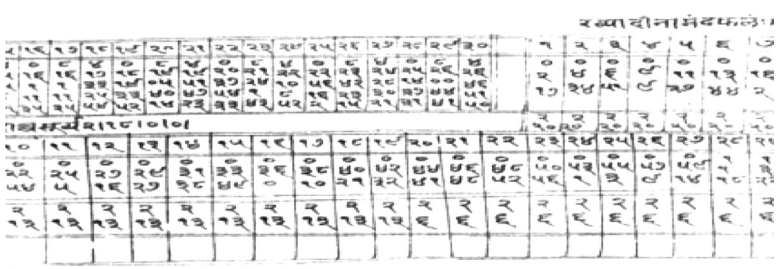


fig. 6.2: *Mandaphala* table of the sun, a folio from the *Karaṇa Kutūhala Sāriṇī*

fig 6.3: Daily motion table of the moon for 30 days, a folio from the *Karaṇa Kutūhala Sāriṇī*

Motion for 28D = $0^R08°56'16''30'''$ by adding all these and subtracting $K = 10^R29°05'50''$ once and then removing the cycles of 12 *rāśi*s

Mean moon = $3^R14°46'48''07''' = 104°46'48''07'''$.

Finding Mandocca of the Moon

From the tables of *candrocca*:

Motion for 30YP = $2^R09°03'17''49'''$

Motion for 12YP = $1^R06°45'07''13'''$

Motion for 7Y = $9^R10°37'41''12'''$

Motion for 10M = $1^R03°24'29''11'''$

Motion for 28D = $0^R03°07'04''04'''$ by adding all these we get $14^R\,02°57'39''29'''$ and subtracting $K = 4^R\,15°12'59''$ and then by removing the cycles of 12 *rāśī*s.

Mandocca of the moon = $287°44'40''$

Mandakendra (*mk*) = *Mandocca* – Mean Moon

$= 177°02'08'' < 180°$

Bhujā = $2°57'52''$

From *candra manda* tables,

Mandaphala (*MP*) = $0°10'35'' + 0°57'52'' \times 0°68'15'' = 1°16'24''$

True moon = Mean Moon + *MP*

$= 104°46'48'' + 1°16'24''$

True moon = $106°03'12'' = 3^R16°03'12''$.

Table 6.7: True (Nirayaṇa) Longitudes of the Sun, the Moon and the Planets

Heavenly Bodies	*Karaṇa Kutūhala (Mean Sunrise at Ujjain)*	*Karaṇa Kutūhala Sāriṇī (Mean Sunrise at Ujjain)*	*Modern (acc. to Ephemeris) at 5:30 a.m. IST*
Sun	$7^R11°28'03''$	$7^R11°15'03''$	$7^R11°34'10''$
Moon	$3^R14°38'49''$	$3^R16°03'12''$	$3^R15°05'28''$
Mars	$10^R10°56'14''$	$10^R13°26'$	$10^R13°26'$
Mercury	$7^R10°52'42''$	$7^R10°42'21''$	$7^R10°06'$
Jupiter	$7^R14°27'40''$	$7^R14°27'22''$	$7^R10°12'$
Venus	$6^R01°17'20''$	$6^R01°27'13''$	$6^R03°27'$
Saturn	$8^R10°54'06''$	$8^R10°51'51''$	$8^R13°23'$

Similarly, we can find the true positions of the five planets by finding their mean planet first, later first *manda* corrected planet followed by first *śīghra* corrected planet again *manda* correction to the mean planet followed by second *śīghra* correction that gives the true position of the planet.

By using both the *Karaṇa Kutūhala* and the *Karaṇa Kutūhala Sāriṇī* the true longitudes of heavenly bodies, the sun, the moon and that of five planets are found and the same are compared with that of ephemerical values and it is listed in Table 6.7.

Conclusion

We have discussed the procedures of both the *Karaṇa Kutūhala* and the *Karaṇa Kutūhala Sāriṇī* to find the true longitudes of the sun, the moon and the planets in the above sections. We can notice that the values obtained from both the *Karaṇa Kutūhala* and the *Karaṇa Kutūhala Sāriṇī* are almost same but when it is compared with ephemerical values slight variation is found hence the revision in the parameters is required. On revision of daily motion of the sun, the moon and the planets we can obtain the position that matches with the ephemerical values. By using tables computation can be made easier, especially for the traditional almanac makers for the compilation of annual almanacs.

References

Mishra, Satyendra, 1989, *Karaṇakutūhalam of Bhāskarācārya II*, with *Gaṇaka-kumuda-kaumudī* comm. of Sumatiharṣa, Hindi tr., Bombay: Krishnadass Academy.

Montelle, Clemency, 2014, "From Verses in Text to Numerical Table: The Treatment of Solar Declination and Lunar Latitude in Bhāskara-II's Karaṇakutūhala and the Related Tabular Work, the Brahmatulyasāriṇī", *Ganita Bharati,* vol. 36, no. 1-2 (2014).

Montelle, Clemency and Kim Plofker, 2015, "The Transformation of a Handbook into Tables: The Brahmatulyasāriṇī, and the Karaṇakutūhala of Bhāskara", *SCIAMVS,* 16: 1-34.

Plofker, Kim, 2014, "Sanskrit Astronomical Tables: The State of the Field", *Indian Journal of History of Sciences*, 49(2): 87-96.

Rao, S. Balachandra and S.K. Uma, 2008, *Karaṇakutūhalam of Bhāskarācārya II*: *An English Translation with Notes and Appendices*, NewDelhi: INSA.

7

Hemāṅgada Ṭhakkura's Grahaṇamālā Eclipses from 1620 to 2708 CE

V. Vanaja
M. Shailaja
S. Balachandra Rao

Abstract: Eclipses are the natural phenomena which frequently occur in nature. The event of an eclipse plays an important role in the religious life of mankind and also occupies an important place in the classical Siddāntic astronomy in India. Hemāṅgada Ṭhakkura (Śaka 1530-90) has listed the data of circumstances of both solar and lunar eclipses visible in India from 1620 to 2708 CE in his text *Grahaṇamālā*. This text gives the circumstances of around 1,437 eclipses for a long period of 1,089 years. The listed data is based on the solar and lunar calendrical terms such as *śaka*, *dyuvṛnda* (*ahargaṇa*), i.e. number of days since the beginning of that solar year, instans of full moon and new moon, weekday, *nakṣatra*, *yoga*, half-duration, beginning and end time of the eclipse. In the present paper we have critically studied the text *Grahaṇamālā* and the given circumstances of the eclipses are verified by using different Indian classical Siddāntic text procedures. We have also compared the results with modern ones.

Keywords: Lunar eclipse, solar eclipse, *śaka*, *dyuvṛnda* (*ahargaṇa*), *nakṣatra*, *yoga*, instances of full moon and new moon, half-duration, beginning (*sparśa*), ending (*mokṣa*).

Introduction

THE text *Grahaṇamālā* was written by Mahāmahopādhyāya Hemāṅgada Ṭhakkura (Śaka 1530-90) and it was edited by Pandit Shri Vrajkishore Jha, a Professor of Kameshwar Singh Darbhanga Sanskrit University, Kameshwar Nagar, Darbhanga, in the year 1983 CE. In this book he has listed 1,437 eclipses among them 399 solar eclipses and 1,038 lunar eclipses starting from Śaka 1542 (1620 CE) to 2630 (2708 CE). The contents of the book is as follows (problem identification):

1. Śaka
2. *Dyuvṛnda* (*ahargaṇa*), i.e. number of days since the beginning of that solar year.
3. Instant of full moon and new moon.
4. Nakṣatra from Aśvinī, etc. for eclipse day.
5. *Yoga* (*Viṣkambha* in *Daṇḍas,* etc).
6. Weekday; number of elapsed days in the corresponding solar month.
7. Name of the lunar month and half-duration of the eclipse.
8. Beginning of the eclipse (*sparśa kāla*).
9. End of the eclipse (*mokṣa kāla*).
10. Moon's latitude (South or North).

To verify the given data of the eclipses we used the Indian Siddhāntic procedures. According to the data to get the eclipsed date and its circumstances we should have the information regarding our Indian calendrical system of both lunar as well as solar.

Calendar Analysis

The text *Grahaṇamālā* gives the data in the following format:

★ ॐ नमः श्रीसूर्याय ★

महामहोपाध्याय-हेमाङ्गदठक्कुर-विरचिता

ग्रहणमाला

खण्डवलाकुलतरणे - गोपालादाप यं गौरी ।
हेमाङ्गदः स तनुते पञ्जीं राहूपरागस्य ॥१॥

शाके १५४२ पूर्णिमा दं ६०।०० द्यु वृन्द ६७ ज्येष्ठा ३१।५०.... .... — ...
र ५ ज्येष्ठी स्थित्यर्धं... ... ... — ... . — ... . —
शाके १५४२ द्यु वृन्द २४५ पूर्णिमा ४३।१९ रोहिणी १२।८ सा १२।४० बु सं
मार्गी स्थित्यर्धं ४।२७ स्पर्श ३८।५२ मुक्ति ४७।४६ ।
शाके १५४३ द्यु वृन्द ४३ अमावास्या २२।५७ कृत्तिका १।१२ अ० ००।४२
शु० १४ ज्यैष्ठी स्थित्यर्धं १।४१ स्पर्श २४।१६ मुक्ति २७।४१ ।
शाके १५४४ द्यु वृन्द २०९ अमावास्या १३।३९ स्वाती २।१४ सौ० ४४।२३
वृ० २३ कार्त्तिकी स्थित्यर्धं २।११ स्पर्श ११।२४ मुक्ति १५।४६ ।
शाके १५४५ द्यु वृन्द पूर्णिमा ... ... — आश्विनी... ... ... — ...
शाके १५४५ द्यु वृन्द ३४५ अमावास्या ११।८ उत्तरभाद्र ३४।५७ शु० ००।०९
मं ११ चैत्री स्थित्यर्धं १।११ स्पर्श ७।५० मुक्ति १०।१२ ।
शाके १५४६ द्यु वृन्द ..... पूर्णिमा ४४।१० हस्त २९।३७ व्या० ४४।० बु० २६
चैत्री स्थित्यर्धं ४।४६ स्पर्श ३९।४६ मुक्ति ४८।३४ ।
शाके १५४६ द्यु वृन्द १७१ पूर्णिमा ४४।२६ उत्तरभाद्र ५१।२६ वृ० २७।२७
वृ० १५ आश्विनी स्थित्यर्धं ४।४६ स्पर्श ४३।२७ मुक्ति ५२।५९ ।

Here Śaka era starts from 78 CE and it is very widely used for both solar and lunar calendars. The data of these eclipses start from Śaka 1542 and end in Śaka 2630 which is equivalent to the Christian calendar year from 1620 to 2708 CE. Specified *tithi* name and also its time in terms of *daṇḍa* (unit for time in that period, i.e. 1 civil day = 60 *daṇḍa*s) one for lunar eclipse, i.e. *pūrṇimā* and other one for solar eclipse, i.e. *amāvāsyā*. For example, the given Śaka is 1542, add 78 to this to get the Christian year, i.e 1542 + 78 = 1620 CE.

Dyuvr̥nda

Dyuvr̥nda is nothing but the number of days elapsed from a particular year from one *meṣa-saṅkramaṇa* to another *meṣa-saṅkramaṇa*. The sun enters into Meṣa *rāśi* is known as *meṣa-saṅkramaṇa* in solar year. Solar year is the time taken by the sun to go around the ecliptic once with reference to the fixed stars. The solar year starts when the sun enters the constellation Meṣa. In the current century this is around April (14 or 15). The solar year is divided into twelve solar months. Using Siddhāntic procedure for the above-cited data, the *meṣa-saṅkramaṇa* of 1542 Śaka falls on 7 April 1620 CE that is on Sunday (this is in sixteenth century). To get this result, we have used the Kali epoch as the midnight between 17 or 18 February 3102 BCE (Julian) and the weekday is considered as Friday, so assumed that to get the *dyuvr̥nda* he considered as the epoch as *meṣa-saṅkramaṇa* of a particular year.

Here we added *dyuvṛnda* to *meṣa-saṅkramaṇa* to get the eclipsed date, for first example, *dyuvṛnda* is 67 it falls on 14 June 1620 CE, Sunday but that day eclipse did not occur and actual eclipse occurred on 15 June 1620 CE, Monday. This we verified using modern and Siddhāntic procedures and also with the NASA data. After *dyuvṛnda* he mentioned instant of full moon or new moon. According to Siddhāntic procedure to get instant of full moon or new moon, we need the true positions of the sun and the moon and their daily motions from the *iṣṭa kāla* of that day it can be calculated as

$$I = \left(\frac{(\text{True Sun} + 180) - \text{True Moon}}{(\text{MDM} - \text{SDM})} \right) \times 24^h,$$

where *MDM* = daily motion of the moon; *SDM* = daily motion of the sun.

He mentions that every lunar eclipse occurs on a full moon day (*pūrṇimā*) and solar eclipse on a new moon day (*amāvāsyā*), and the time unit as *daṇḍa* that is considered as 1 civil day = 60 *daṇḍa*s which is equivalent to 24 hours. Therefore 1 hour = 2.5 *daṇḍa*s. Then he has given the time of *nakṣatra* and *yoga* of the eclipsed day in *daṇḍa*s. The "asterism", one of the 27 divisions of the zodiac from Aśvinī to Revatī, occupied by the *nirayaṇa* moon is mentioned. *Yoga* is the sum of the *nirayaṇa* longitudes of the sun and the moon is divided into 27 equal divisions. There are 27 *nirayaṇa yoga*s. They are *viṣkambha, prīti, āyuṣmān, …, indra, vaidhṛta*. He has mentioned the name of *nakṣatra* and *yoga* using the lunar month (a period from one new moon to the next new moon). The lunar calendar of our Indian system of lunar months are Caitra, Vaiśākha, …, Phālguna. For the calculation of these *pañcāṅga* elements, refer Balachandra Rao's book *Indian Astronomy: Concepts and Procedures* (2014).

In this text we found another important data in the form of specific number and short week day name. Here the given number belongs to the number of days elapsed in that solar month of the luni–solar calendar.

Example:

शाक १२३ द्युवश्न्द 20 पूर्णिमा ४५।५३ स्वाती ४४।२१ शि १२।६ शु २०
वैशाखी स्थित्यद्र्ध १।२७ स्पर्श ४४।२६ मुक्ति ४७।२० शर सौम्य।।

śaka 1823 *dyuvṛnda* 20 *pūrṇimā* 45/53 *svātī* 44/21 *śi* 12/6 *śu* 20 *vaiśākhi sthityarddha* 1/27 *sparśa* 44/26 *mukti* 47/20 *śara saumya* ।।

Year = 1823 = + 78 = 1901 CE, in this particular year, the date of *meṣa-saṅkramaṇa* was fallen on 13/04/1901 to this add 20 days to get the actual eclipse date, i.e. 3 May 1901 CE and *tithi* was *pūrṇimā* the running *nakṣatra* and *yoga* of eclipsed day were *svātī* and *siddhi* respectively. In this data *śu* corresponds to weekday *śukravāra* (Friday), the number 20 corresponds to the elapsed days in solar month, i.e. Meṣa, and lunar month is Vaiśākha.

To predict the solar and lunar months of the year, *fig*. 7.1 will be useful. It is consisted of twelve lunar and solar months, the beginning of the lunar year is at the instant of the new moon (i.e. final moment of *amāvāsyā* of the previous lunar month) occurring in the course of the solar Caitra (i.e. when the sun is in Mīna *rāśi*). The second month of the lunar calendar, viz. Vaiśākha, starts at the following new moon and so on. In the chart, N_0, N_1, N_2, etc. refer to new moons.

Here we have considered the computation of lunar eclipse date and compared that with of the Modern NASA tables and Siddāntic procedures. We have obtained an algorithm using Scilab software to compute lunar eclipse for the dates given in the text *Grahaṇamālā*.

Comparison of lunar eclipse circumstances according to the *Grahaṇamālā*, Siddhāntic text and modern techniques for the date 31 January 2018. The *Grahaṇamālā* data is as follows:

शाके १९३९ द्युवश्न्द २९२ पूर्णिमा ३२।३२ पुष्य २९।४१ प्री ००।३९ बु १६
माघी स्थित्यर्द्ध ४।२५ स्पर्श २८।०७ मुक्ति ३६।५७ शर सौम्य।।

śāke = 1939 *dyuvṛnda* 292 *pūrṇimā* 32/32 *puṣya* 29/41 *prī* 00/39 *bu* 16 *Māghī sthityarddha* 4/25 *sparśa* = 28/07 *mukti* = 36/57 *śara saumya* ।।

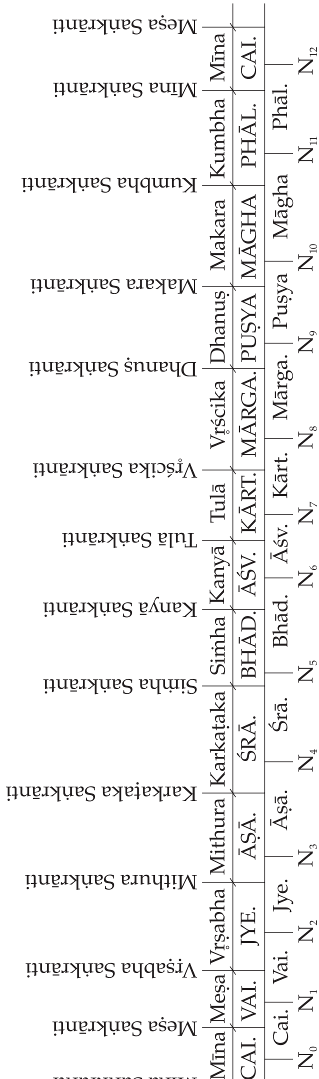


fig. 7.1: Prediction of the solar and lunar months

Computation of Lunar Eclipse by Indian Siddhāntic Procedure

Using the Indian Siddhāntic procedure and its terms like true positions of the sun, the moon and daily motions we constructed the following algorithm to compute lunar eclipse and called it as Improved Siddhāntic Procedure (ISP).

Example: Lunar eclipse on 31 January 2018, Wednesday.

Instant of opposition is $18^h58^m57^s$ (IST).

At the instant of opposition

True sun: 286°56'33"; True Moon: 99°06'13"; Rāhu: 110°49'38".

Sun's daily motion, SDM: 1°.014722

Moon's daily motion, MDM: 14°.968611

(i) Moon's latitude (*candra śara*) = β = 308' × sin ($M - R$) = – 0.296808 = – 17'.808384.

(ii) Moon's angular diameter (*candra bimba*) = *MDIA* = $2\frac{[939.6+(61.1)\cos GM]}{60}$ in minutes of arc where *GM* is the moon's anomaly (*mandakendra*) measured from its perigee and it is given by

$$GM = 134°.9633964 + 13°.06499295T + \cdots,$$

where *T* be the number of days completed since the epoch 1 January 2000, noon (*GM*T), i.e. $18^h58^m57^s$ (IST). *JD* for this particular date, i.e. 31 January 2018 is 2458150.

JD for $1^h28^m = (18^h58^m17^h30^m) = \frac{1^h28^m}{24^h} = 0.061111$ days.

$\therefore$ the days from the epoch (1 January 2000, noon *GM*) is

$T = JD$ for 31 January 2018 – *JD* for 1 January 2000.

$T = 2458150.061111 - 2451545 = 6605.0611.$

Using the value of *T* in *GM* it obtains the value

$GM = 134°.9633964 + 13°.06499295T + \ldots = 30°.029387$

$\therefore$ *MDIA* = 33'.083283.

GS = The sun's mean anomaly from its perigee

$$= 357°.529092 + 0°.985600231\ T = 27°.4795.$$

(iii) Diameters of the earth's shadow (*chāyā bimba*):

$SHDIA = 2\ \frac{[2545.4 + 228.9\cos GM - 16.4\cos GS]}{60}$ in minutes of arc

SHDIA = 90'.967493

True daily motions of the sun and the moon:

vyarkendu sphuṭa nāḍī gati, *VRKSN* = (*MDM* - *SDM*) per *nāḍī* i.e. $VRKSN = \frac{(MDM - SDM)}{60} = 13'.953889$

Note: One day = 60 *nāḍī*s; 1 *nāḍī* = 60 *vināḍī*s = 24 minutes

(iv) *Bimba yogārdham* = $D = \frac{(MDIA - SHDIA)}{2} = 62'.025388.$

(v) *Bimba viyogārdham* = $D' = \frac{SHDIA - MDIA}{2} = 28'.942105.$

(vi) *Sphuṭa śara* = $\beta' = \beta \times \left(1 - \frac{1}{205}\right) = -0.295360 = -17'.721513,$ where β is the moon's latitude from step (i) above.

(vii) *MDOT*, $\dot{m} = VRKSN \times \left(1 + \frac{1}{205}\right) = 14'.021968.$

(viii) If $|\beta'| < D'$, then lunar eclipse occurs. If $|\beta'| < D'$, then the eclipse is total.

In this case $|\beta'| < D'$, i.e. $17'.7216 < 28'.942105$. Hence, the eclipse is total.

(ix) *VīRāhu Candra*, *VRCH* = (True Moon – True Rāhu)

$$= -3^0.314671 = -3^0 18' 53''.$$

(x) Calculate: $COR = \frac{|\beta'| \times 59}{10 \times \dot{m}}$ *vināḍī*s.

(a) If *VRCH* is in an odd quadrant (i.e. I or III), then subtract the above value *COR* from the instant of opposition to get the instant of the middle of the eclipse.

(b) If *VRCH* is in an even quadrant (i.e. II or IV), then add the above value *COR* to the instant of opposition to get the instant of the middle of the eclipse.

In the current example, $COR = \frac{|\beta'| \times 59}{10 \times \dot{m}} = 0'.049711.$

Now, $VRCH = 357^\circ 18' 53''$. Since $VRCH > 270^\circ$, i.e. *VRCH* is in IV quadrant (even), the above value is additive from the instant of opposition.

$\therefore$ Middle of the eclipse = Instant of opposition + COR

$$= 18^h 58^m 57^s + 0^h 2^m 59^s$$

$$= 19^h 01^m 56^s$$

(xi) Half-duration of the eclipse (*Sthiti*)

$HDUR = \frac{\sqrt{D^2 - (\beta')^2}}{\dot{m}} = 4.23905$ *nādi* $= 4.23905 \times \frac{2}{5} = 1^h 41^m 44^s$

(xii) Half-duration of totality (*marda*)

$THDUR = \frac{\sqrt{(D')^2 - (\beta')^2}}{\dot{m}} = 1.631875$ *nādi* $= 1.631875 \times \frac{2}{5}\ 0^h 39^m 10^s$

Summary of the Eclipse	*IST*
Beginning of the eclipse (*sparśa*) = Middle – *HDUR* = $19^h01^m56^s - 1^h41^m44^s$	
	$= 17^h20^m12^s$
(1) Beginning of totality (*sammīlana*) = Middle – *THDUR* = $19^h01^m56^s - 0^h39^m10^s$	
	$= 18^h22^m46^s$
(2) Middle (*madhya*) = Instant of full moon + *COR*	$= 19^h01^m56^s$
(3) End of totality (*unmīlana*) = Middle *THDUR* = $19^h\ 01^m\ 56^s + 0^h39^m10^s$	
	$= 19^h41^m06^s$
(4) End of the eclipse (*mokṣa*) = Middle *HDUR* = $19^h\ 01^m\ 56^s + 1^h41^m44^s$	
	$= 20^h43^m40^s$

$$Pramāṇam \text{ (Magnitude)} = \frac{D-(\beta')^2}{MDIA} = 1.339162.$$

We are comparing these results with the *Grahaṇamālā* data, modern procedure and also with NASA. We can identify the variation of the range in IST of 2 minutes in our ISP.

Now, we consider a few data related to solar eclipses as given in the *Grahaṇamālā*.

शाके १५४३ द्युवश्न्द ४३ अमावास्या २२।५७ कृत्तिका ११।२ अ० ००।४२ शु० १४ ज्यैष्ठी स्थित्यर्ध १।४१ स्पर्श २४।१६ मुक्ति २७।४१।

śāke 1543 *dyuvṛnda* 43 *amāvāsyā* 22/57 *kṛttikā* 11/2 *am* 00/42 *śu* 14 *jyaiṣṭhī sthityardha* 1/41 *sparśa* 24/19 *mukti* 27/41.

Table 7.1: Lunar Eclipse Circumstances in IST with Different Procedures

	Grahaṇamālā	*ISP*	*Modern*	*NASA*
Beginning of the eclipse (*sparśa*)	$17^h14^m48^s$	$17^h20^m12^s$	$17^h20^m32^s$	$17^h18^m27^s$
Beginning of totality (*sammīlana*)	—	$18^h22^m46^s$	$18^h23^m29^s$	$18^h21^m47^s$
Middle (*madhya*)	$19^h00^m48^s$	$19^h01^m56^s$	$19^h01^m09^s$	$18^h59^m49^s$
End of totality (*unmīlana*)	—	$19^h41^m06^s$	$19^h38^m49^s$	$19^h37^m51^s$
End of the eclipse (*mokṣa*)	$20^h46^m48^s$	$20^h43^m40^s$	$20^h41^m46^s$	$20^h41^m11^s$

शाके १५४४ द्युवश्न्द २०९ अमावास्या १३।३९ स्वाती २।१४ सौ० ४४।२३ वश० २३ कार्त्तिकी स्थित्यर्ध २।११ स्पर्श ११।२४ मुक्ति १५।४६।

śāke 1544 *dyuvṛnda* 209 *amāvāsyā* 13/39 *svāti* 2/14 *sau* 44/23 *bṛ* 23 *kārttikī sthityardha* 2/11 *sparśa* 11/24 *mukti* 15/46.

शाके १५४५ द्युवश्न्द ३४५ अमावास्या ११।८ उत्तरभाद्र ३४।५७ शु० ००।०९ मं ११ चैत्री स्थित्यर्ध १।११ स्पर्श ७।५० मुक्ति १०।१२।

śāke 1545 *dyuvṛnda* 345 *amāvāsyā* 11/8 *uttarābhādra* 34/57 *śu* 00/09 *ma* 11 *caitrī sthityardha* 1/11 *sparśa* 7/50 *mukti* 10/12

शाके १९३२ द्युवश्न्द २६४ अमावास्या २३।३१ पूर्वाषाढ़ ५३।३३ ध्रु २५।२ मं १९ पौष स्थित्यर्द्ध १।३० स्पर्श २२।३२ मुक्ति २५।३२ शर सौम्य।।

śāke 1932 *dyuvṛnda* 264 *amāvāsyā* 23/31 *pūrvāṣādha* 53/33 *dhru* 25/2 *ma* 19 *pauṣī sthityardha* 1/30 *sparśa* 22/32 *mukti* 25/32 *śara sanmya*

शाके १९३७ द्युवश्न्द ३२९ अमावास्या ००।१२ पूर्वभाद्र ४०।२४ सा १९।१ बु २५ फाल्गुनी स्थित्यर्द्ध २।१९ स्पर्श ५७।४१ मुक्ति २।१९ शर याम्य।।

śake 1937 *dyuvṛnda* 329 *amāvāsyā* 00/12 *pūrvabhādra* 40/24 *sā* 19/1 *bu* 5 *phālguni sthityardha* 2/19 *sparśa* 57/41 *mukti* 2/19 *śara yāmya*

शाके १९४१ द्युवश्न्द २५५ अमावास्या ११।५४ मूल २८।९ वश ४२।१५ वश ९ षौषी स्थित्यर्द्ध २।२९ स्पर्श ८।४५ मुक्ति १३।४३ शर याम्य।।

śake 1941 *dyuvṛnda* 255 *amāvāsyā* 11/54 *mūla* 28/9 *bṛ* 42/15 *bṛ* 9 *pauṣī sthityardha* 2/29 *sparśa* 8/45 *mukti* 13/43 *śara yāmya*.

Computation of Solar Eclipse by Indian Siddhāntic Procedure

Using the *Improved Siddhāntic procedure*, we verify one of the data given in the *Grahaṇamālā*. Consider an eclipse of data given in the year 2016 CE.

According to the above data, the eclipsed date occurred on 9 March 2016 is considered as an example.

Example: Solar eclipse for the world in general was on 9 March 2016.

(i) For the given date at 5.30 a.m. (IST) from IAE.

1. True longitude of the sun = 324°45'58"

2. True longitude of the moon = 323°39'38"
3. True longitude of the Rāhu = 147°42'
4. Sun's true daily motion (*SDM*) = 59'59"
5. Moon's true daily motion (*MDM*) = 14°58'04".

(ii) Instant of conjunction:

$$5^h\,30^m\ \frac{\text{Difference in true longitude of the sun and the moon}}{\text{Difference in their daily motions}} \times 24$$

$= 5^h30^m + (04'44.94'' \times 24) = 5^h30^m + 1^h53^m58^s$

$= 7^h23^m58^s$ a.m. (IST).

Difference in time = instant of conjunction – 5.30 a.m.

$= 1^h53^m58^s$.

(iii) True longitudes at the instant of conjunction:

True longitude of the sun = 324°50'43".

True longitude of the moon = 324°50'43".

Longitude of the Rāhu = 147°41'41".

(iv) Find anomalies (*GM* and *GS*) of the moon and the sun (from perigee).

Let T be the number of Julian days completed since the epoch 1 January 2000, noon (GMT), i.e. 17^h30^m.

Using the *ahargaṇa* tables, find the *JD* for 9 March 2016.

In this example

T = No. of *JD* for 9 March 2016 at $7^h23^m58^s$ – *JD* at epoch

$T = 2457356.57914352 - 2451545 = 5811.57914352$.

$GM = 134^\circ.9633964 + 13^\circ.06499295T + \ldots + = 103^\circ.186778$.

$GS = 357^\circ.5291092 + 0^\circ.985600231T + \ldots + = 325^\circ.422838$,

where *GM* and *GS* are the moon's and the sun's anomaly from its perigee respectively.

(v) The true angular diameters of the sun and the moon are given as *SDIA* and *MDIA* respectively.

$$SDIA = 2\ \frac{\lfloor 961.2 + (16.1) \times \cos GS \rfloor}{60} \text{ in minutes of arc}$$

$= 32'.481871 = 32'28''$.

$MDIA = 2\ \dfrac{[939.6 + (16.1) \times \cos GM]}{60}$ in minutes of arc

$= 30'.855383 = 30'51''.$

(vi) Moon's horizontal parallax is given by

$PAR = \dfrac{[3447.9 + 224.4 \times \cos (GM)]}{60}$ in minutes of arc

$= 54'.157677 = 54'9''.$

The sum of the semi-diameters of the moon and the sun with addition of the moon's parallax is denoted by D.

i.e. $D = PAR \left(\dfrac{MDIA + SDIA}{2}\right) = 85'.826304 = 85'49''.$

The difference of the semi-diameters of the moon and the sun with addition of the moon's parallax is denoted by $D1$.

i.e. $D1 = PAR \left(\dfrac{MDIA + SDIA}{2}\right) = 53.344433 = 53'20''.$

The moon's latitude is $\beta = 308' \sin (M - R)$, where M and R denote the true longitudes of the moon and Rāhu at the instant of conjunction.

Here, $\beta = 15'.339750 = 15'20''.$

The apparent latitude of the moon is given by

$\beta1 = \beta \dfrac{204}{205} 15'.264922 = 15'15''$

In this example, $|\beta1| < D1$, i.e. $15'15'' < 53'20''$. Hence eclipse is total.

(vii) *Vyarkendu nāḍī gati* $= \left(\dfrac{MDM - SDM}{60}\right)$ in minutes $= 13'.968056$

$= 13'58''.$

The apparent rate of motion of *vyarkendu nāḍī gati*, denoted by $\dot{m}$. It is given by $\dot{m} = VRK \times \dfrac{206}{205} = 14'.036192 = 14'2''.$

(viii) Let *Virāhucandra*, $VRCH$ = True Moon – Rāhu = $177^{o}.145238$ $= 177^{o}8'43''.$

Note: If $VRCH < 0$, then add 360^{o} to get the positive value of it.

(ix) The middle of the eclipse: instant of Conjunction Time $\pm$ COR, where, $COR = \dfrac{99 \times |\beta1|}{1000 \times \dot{\underline{m}}} = 2^{m}35^{s}$, called "correction" in *nāḍī*, which is added or subtracted as $VRCH$ is in even and odd quadrant respectively.

Middle of the eclipse = Instant of Conjunction + *COR*

$= 7^h 23^m 58^s + 0^h 2^m 35^s$

$= 7^h 26^m 33^s$a.m. (IST).

(x) Half-interval of the eclipse is given by

$HDUR = \frac{\sqrt{(D)^2 - (\beta 1)^2}}{\underline{\dot{m}}} = 2.406861 = 2^h 24^m 24^s.$

$THDUR = \frac{\sqrt{(D1)^2 - (\beta 1)^2}}{\underline{\dot{m}}} = 1.456626 = 1^h 27^m 24^s.$

The beginning and the end moments of the eclipse as also of totality are obtained as follows:

Summary of the Eclipse

Beginning of the eclipse	:	Middle – *HDUR* = $5^h 02^m 09^s$ (IST).
Beginning of the totality	:	Middle – *THDUR* = $5^h 59^m 09^s$ (IST).
Middle of the eclipse	:	$7^h 26^m 33^s$ (IST).
End of the totality	:	Middle + *THDUR* = $8^h\ 53^m 57^s$ (IST).
End of the eclipse	:	Middle + *HDUR* = $9^h 50^m 58^s$ (IST).

Table 7.2 gives the circumstances of the solar eclipse occurred on 9 March 2016 with data of the *Grahaṇamālā*, *ISP*, Modern and NASA in Indian Standard Time (IST).

From Table 7.2, we can observe the circumstances of the solar eclipse of the *Grahaṇamālā* and the *Sūrya Siddhānta* data which are

Table 7.2: Solar Eclipse Circumstances in IST with Different Procedures

Event	*Grahaṇamālā*	*Sūrya Siddhānta (Bangalore)*	*ISP*	*Modern*
Beginning of the eclipse (*sparśa*)	$5^h 04^m 24^s$	$5^h 21^m 12^s$	$5^h 02^m 09^s$	$4^h 49^m 48^s$
Beginning of totality (*sammīlana*)	—	—	$5^h 59^m 09^s$	$5^h 49^m 43^s$
Middle (*madhya*)	$6^h 04^m 48^s$	$6^h 05^m\ 36^s$	$7^h 26^m 33^s$	$7^h 28^m 19^s$
End of totality (*unmīlana*)	—	—	$8^h 53^m 47^s$	$9^h 06^m 55^s$
End of the eclipse (*mokṣa*)	$6^h 55^m 36^s$	$6^h 49^m 00^s$	$9^h 50^m 58^s$	$10^h 06^m 50^s$

comparable and computed for a particular place. The *Sūrya Siddhānta* data gives a place called Bangalore. According to the text *Grahaṇamālā*, the given data corresponds to Ujjainī because the author of this text belongs to north India. The ISP and modern procedures give data related to the world in general. However, the middle of the eclipse calculations is comparable to one another with respect to their procedures.

Conclusion

We have discussed the text the *Grahaṇamālā* of Hemāṅgada Ṭhakkura and the data of this text are verified by the Indian Siddhāntic procedures of both the eclipses. These data have variations with a few minutes in IST. The lunar eclipse data differ in their circumstances with 2 minutes for the world in general. The circumstances of the solar eclipse are different for a particular place and the world in general.

References

Montelle, Clemency, 2011, *Chasing Shadows*, Baltimore, MD: Johns Hopkins University Press.

Jha, Ujjwala (ed.), 2008, *Eclipses in Siddhāntas: Improved Procedures, Dimensions of Contemporary Sankskrit Research*, Delhi: New Bharatiya Book Corp.

S. Balachandra, S.H. Uma and Padmaja Venugopla, 2003, "Lunar Eclipse Computation in Indian Astronomy with Special Reference to Grahalāghava, *Indian Journal of History of Sciences* 38(3): 255-76.

Meeus, Jean, 2000, *Mathematical Astronomy Morsels*, Virginia: Willmann-Bell.

Rao, S. Balachandra and Padmja Venugopal, 2008, *Eclipses in Indian Astronomy*, Bangalore: Bhavan's Gandhi Centre of Science and Human Values.

Rao, S. Balachandra and Padmaja Venugopal, 2009, *Lunar Eclipse Computations and Solar Eclipse Computations*, National. Workshop on Computation of Planetary Positions and Almanac, pp. 7-28 and pp. 70-104, Hyderabad: I-SERVE.

Rao, Balachandra S., 2014, *Indian Astronomy*: *Concepts and Procedures*, Bengaluru: M.P. Birla Institute of Management.

Shylaja, B.S. and H.R. Madhusudan, 1999, *Eclipse*: *A Celestial Shadow Play*, Hyderabad: University of Hyderabad Press.

8

Makarandasāriṇī
Some Special Features

S.K. Uma
Padmaja Venugopal
K. Rupa
S. Balachandra Rao

Abstract: In our Indian society, for both civil and religious purposes, the annual astronomical almanacs called *pañcāṅga*s are almost a necessity. Compilation and use of annual *pañcāṅga*s are a socio-religious necessity in our Hindu society. These annual *pañcāṅga*s are compiled using traditional astronomical tables called *sāriṇī*s, *padaka*s or *koṣṭhaka*s. These tables, in turn, are constructed based on classical texts like the *Sūryasiddhānta*, *Āryabhaṭīyam*, *Brahmasphuṭa-siddhānta* and *Grahalāghava*. These texts have given rise to different *siddhānta pakṣa*s (schools of astronomy) called (i) Saurapakṣa, (ii) Āryapakṣa, (iii) Brāhmapakṣa, and (iv) Gaṇeśapakṣa.

These different *pakṣa*s conformed to the parameters and procedures respectively of the *Sūryasiddhānta* and *Āryabhaṭīyam* of Āryabhaṭa I (476 CE), *Brahmasphuṭasiddhānta* of Brahmagupta (628 CE) and *Grahalāghava* of Gaṇeśa Daivajña (1520 CE). Since the direct application of the major texts is cumbersome and tedious for day-to-day positions of heavenly bodies, the *pañcāṅga*s are compiled annually based on *sāriṇī*s (tables) of different *siddhānta pakṣa*s.

Among the tables of Saurapakṣa, the most popular is the *Makarandasāriṇī*. These tables with explanatory *śloka*s are composed by Makaranda, son of Ānanda of Kāśī in Śaka 1400 (1478 CE). As compared to other Indian astronomical tables the *Makarandasāriṇī* has some unique and special features:

i. Determination of *ahargaṇa* in the sexagesimal system.

ii. For obtaining the true position of the *tārāgraha*s, in other systems the *manda* and *śīghra* equations are generally applied in four stages. But this procedure is reduced to only three significant stages, viz. (a) half *śīghra*, (b) *manda*, and (c) full *śīghra*. In this case the usual half *manda* and full *manda* corrections are combined in a mathematically justified manner.

iii. In the computations of lunar and solar eclipses, the angular diameters of the sun, the moon and the earth's shadow are obtained from the moon's *nakṣatramāna* (duration of *nakṣatra*) and the sun's *saṅkrānti*.

Keywords: *Ahargaṇa*, *Makarandasāriṇī,* Saurapakṣa. Makaranda, *śīghra, manda*, angular diameters, *bimba*s, *nakṣatrabhoga*, *Sūryasiddhānta.*

Ahargaṇa

Literally the word *ahargaṇa* means "heap of days". According to Siddhāntas, it is the number of mean civil days elapsed since a chosen epoch at midnight or mean sunrise for the Ujjain meridian. This meridian passes through Laṅkā, supposedly on the equator. The calculation of *ahargaṇa* depends on the calendar system it follows. The traditional Hindu calendar follows both luni-solar and solar systems. The former is pegged on to the latter through intercalary months. In the present paper, the procedures for finding *ahargaṇa* have been presented in detail with concrete examples.

THE GENERAL PROCEDURE FOR FINDING AHARGAṆA

The process of finding *ahargaṇa* essentially consists of the following steps:

i. Convert the solar year elapsed (since the epoch) into lunar months.

ii. Add the number of *adhikamāsa*s during that period to give the actual number of lunar months that have elapsed up to the beginning of the current lunar year.

iii. Add the number of elapsed lunar months in the given year.

iv. Convert these actually elapsed number of lunar months into *tithi*s (by multiplying it by 30).

v. Add elapsed number of *tithi*s in the current lunar month.

vi. Subtract the *kṣaya dina*s and finally convert the elapsed number of *tithi*s into civil days.

Note: While finding *adhikamāsa*s, if an *adhikamāsa* is due after the lunar month of the current year, then 1 is to be subtracted from the calculated number of *adhikamāsa*s. This is because an *adhikamāsa* which is yet to come in the course of current year would have already been added.

AUDĀYIKA AND ĀRDHARĀTRIKA SYSTEMS

In Indian astronomical texts, the Kali-Yuga is said to have started either at the mean sunrise on 18 February 3102 BCE or at the midnight between 17 and 18 February 3102 BCE. Accordingly, the corresponding systems are called respectively Audāyika (sun rise system) and Ārdharātrika (midnight system).

Interestingly, even the important astronomical parameters are somewhat different in the two systems. In fact, the earliest available systematic text, the *Āryabhaṭīyam* of Āryabhaṭa I (b.476 CE) belongs to the Audāyika system. It is believed that Āryabhaṭa wrote another text – popularly described as the *Āryabhaṭa Siddhānta* which belongs to the Ārdharātrika system. Again, the earliest text of Ārdharātrika system available and popular is the *Khaṇḍakhādyaka* of Brahmagupta (b.598 CE).

TO FIND AHARGAṆA SINCE THE KALI EPOCH

Before evolving a working procedure for finding the Kali *ahargaṇa*, we shall list some useful data for the purpose according to the *Sūryasiddhānta*. In a *mahāyuga* of 432×10^4 years, we have

(i) Number of sidereal revolutions of the moon: 57,753,336

(ii) Number of revolutions of the sun : 4,320,000

(iii) Number of lunar months in a *mahāyuga* of 432×10^4 years given by (i) – (ii) : 53,433,336

Number of *adhikamāsa*s in a *mahāyuga*

= No. of lunar months – (12 × number of solar years)

= 53,433,336 – (12 × 4,320,000)

= 53,433,336 – 51,840,000

= 1,593,336.

Suppose we wish to find *ahargaṇa* for the day on which x luni-solar years, y lunar months and z lunar *tithi*s have elapsed. Then the number of *adhikamāsas* in x completed solar years is given by

$$x_1 = \text{INT}\left[(x)\times\left(\frac{1,593,336}{4,320,000}\right)\right]$$

where INT (i.e. integer value) means only the quotient of the expression in the square brackets is considered.

Now, since in the given luni-solar year, y lunar months and z *tithi*s have elapsed, we have

No. of lunar months elapsed since the epoch

$$= 12x + x_1 + y + \frac{z}{30},$$

where the number of elapsed *tithi*s z is converted into a fraction of a lunar month. The average duration of a lunar month is 29.530589 days. Therefore, the number of civil days N^1 elapsed since epoch:

$$N^1 = \text{INT}\left[\left(12x + x_1 + y + \frac{z}{13}\right)\times 29.530589\right].$$

Here also only the integer part of the expression in the square brackets is considered.

Since in our calculations we have considered only mean duration of a lunar month, the result may have a maximum error of 1 day. Therefore, to get the actual *ahargaṇa* N, addition or subtraction of 1 to or from N^1 may be necessary.

This is decided by the verification of the weekday. The tentative *ahargaṇa* N^1 is divided by 7 and the remainder is expected to give the weekday counted from the weekday of the chosen epoch. For example, the epoch of Kali-Yuga is known to have been a Friday.

Therefore, when N^1 is divided by 7, if the remainder is 0, then the weekday must be a Friday, if 1 then Saturday, etc. However, if the calculated weekday is a day earlier or later than the actual weekday, then 1 is either added to or subtracted from N^1 so as to get the calculated and actual weekday the same. Accordingly, the actual *ahargaṇa* $N = N^1 \pm 1$.

It is important to note that the method described above is a simplified version of the actual procedure described variously by the Siddhāntic texts.

Note: While finding the number of *adhikamāsa*s x_1 in the above method if an *adhikamāsa* is due after the given lunar month in the given lunar year, then subtract 1 from x_1 to get the correct number of *adhikamāsa*s.

Example: Find Kali *ahargaṇa* corresponding to Caitra *kṛṣṇa trayodaśī* of Śaka year 1913 (elapsed), i.e. for 12 April 1991.

Number of Kali years = 3179 + 1913 = 5092, since the beginning of the Śaka era, i.e. 78 CE, corresponds to 3,179 years (elapsed) of Kali-Yuga.

$$\therefore \text{ Adhikamāsas in 5,092 years} = (1{,}593{,}336/4{,}320{,}000) \times 5{,}092 = 1{,}878.0710.$$

Taking the integral part of the above value, $x_1 = 1{,}878$.

Now, an *adhikamāsa* is due just after the Caitra *māsa* under consideration. Although the *adhikamāsa* is yet to occur, it has been included already in the above value of x_1. Therefore, the corrected value of x_1 is $1{,}878 - 1 = 1{,}877$.

Since the month under consideration is Caitra, the number of lunar months elapsed in the lunar year, $y = 0$. The current *tithi* is *trayodaśī* of *kṛṣṇa pakṣa* so that the elapsed number of *tithi*s is 15 + 12 = 27, i.e. $z = 27$.

$$\therefore \text{ Number of lunar months completed} = (5{,}902 \times 12) + 1{,}877 + 0 + 27/30 = 62{,}981.9$$

The number of civil days $N^1 = \text{INT}\,[62{,}981.9 \times 29.530589]$.

$$= \text{INT}\,[1{,}859{,}892.603] = 1{,}859{,}892.$$

Now, dividing N^1 by 7, the remainder is 6; counting 0 as Friday, 1 as Saturday, etc. the remainder 6 corresponds to Thursday. But, from the calendar, 12 April 1991 was a Friday. Therefore, we have the actual *ahargaṇa* $N = N^1 + 1 = 1{,}859{,}893$ since the Kali epoch.

AHARGAṆA ACCORDING TO MAKARANDASĀRIṆĪ

The author of the *Makarandasāriṇī* has incorporated many changes to yield better results during his time. He has given *ahargaṇavallī* table for computing *ahargaṇa* for the given day in sexagesimal system by expressing it in units called *rāśī, aṁśa, kalā* and *vikalā*s. The *adhikamāsa* concept of a lunar calendar is incorporated in the tables of *ahargaṇavallī* in such a way that finding *ahargaṇavallī* from the *Makarandasāriṇī* tables is easier when compared to the procedure for obtaining *ahargaṇa* from other related astronomical texts belonging to Saurapakṣa.

The *Ahargaṇavallī* expressed in *rāśī, aṁśa*, *kalā* and *vikalā*s is equivalent to *ahargaṇa* days expressed as a sum of power of 60. The *Makarandasāriṇī ahargaṇa* is counted from the beginning of Kali-Yuga, Vaiśākha *śuddha pratipath* Friday and is correct to the midnight of the central meridian.

Remark: At the beginning of Kali-Yuga, i.e. at the midnight between 17 and 18 February 3102 BCE, all the mean heavenly bodies were at 0° (Meṣa). This means that was the instant of the mean Meṣa Saṅkrānti and also the mean beginning of the lunar month.

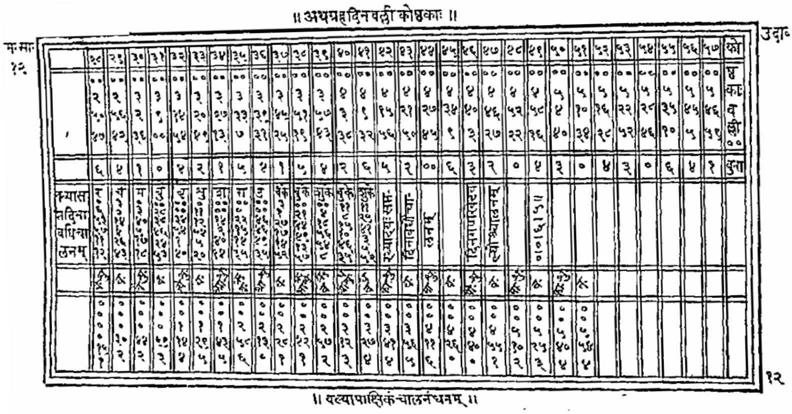
Now, at that moment, *mandakendra* of the sun, $MK = 78^\circ - 0^\circ = 78^\circ$.

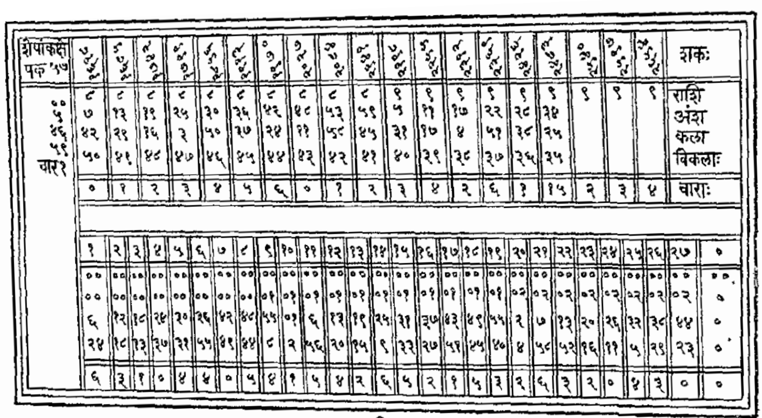
$$\therefore \textit{Mandaphala}, \text{ Equation of the centre} = (14^\circ/2)\,\pi \sin 78^\circ = 2^\circ.17947836$$

by taking the *manda* periphery = 14°

converting into days = $2^\circ.17947836/59'8'' = 2.21142111$ days.

Since the equation of the centre is positive, true Meṣa Saṅkrānti occurs 2 days earlier. That is the beginning of Kali-Yuga, being the end of *amāvāsya* occurs 2 days after the true Meṣa Saṅkrānti. This means it is the beginning of Vaiśākha month. In other words, the beginning of Caitra will have occurred around 19 January 3102 BCE (30 days before).





Tables 8.1, 8.2 and 8.3 (below) give *ahargaṇavallī* for a given date

IMAGE OF AHARGAṆAVALLĪ TABLE

In Table 8.1 *ahargaṇavallī* is given for the tabulated Śaka years with an interval of 57 years, starting from Śaka 1628 up to 2654 [i.e. 1706 CE to 2732 CE]. In the beginning, the first column gives *vallī* for 57 years (called *śeṣāṅka kṣepaka*) in *rāśi*, *aṁśa*, *kalā* and *vikalā*s. Also the last row gives *vāra* (weekday). The table can be generated by adding *vallī* of *kṣepaka* year 57, i.e. 0 | 5 | 46 | 59 and *vāra* 1 to the previous entries correspondingly. This is shown in Example 8.1 below.

Table 8.1 Āhargaṇavallī during Śaka 1628–2654

Śeṣāṅka *Kṣepaka* 57	*Rāśi*	*Aṁśa*	*Kalā*	*Vikalā*	*Vāra*
1628	0	5	46	59	1
1685	8	7	42	50	0
1742	8	13	29	49	1
1799	8	25	3	47	3
1856	8	30	50	46	4
1913	8	36	37	45	5
1970	8	42	24	44	6
2027	8	48	11	43	0
2084	8	53	58	42	1
2141	8	59	45	41	2
2198	9	5	31	40	3
2255	9	11	17	39	4
2312	9	17	4	38	5
2369	9	22	51	37	6
2426	9	28	38	36	0
2483	9	34	25	35	1
2540	9	—	—	—	2
2597	9	—	—	—	3
2654	9	—	—	—	4

Now, 57 solar years = $365.25 \times 57 \approx 20{,}819$ days and 20,819 when multiples of 7 are removed gives *śeṣa vāra* 1 (remaining *vāra* after dividing 20,819 by 7) including leap years.

Dividing 20,819 by 60 we obtain $Q_1 = 346$ and $R_1 = 59$

Now, dividing Q_1 by 60 we get $Q_2 = 5$ and $R_2 = 46$

Dividing Q_2 by 60 we get $Q_3 = 0$ and $R_3 = 5$

Dividing Q_3 by 60 we get $Q_4 = 0$ and $R_4 = 0$.

Thus, *vallī* corresponding to 57 years (*kṣepaka*) = $R_4 \mid R_3 \mid R_2 \mid R_1$ = $0 \mid 5 \mid 46 \mid 59$ and *vāra* = 1.

Table 8.2 gives *ahargaṇavallī* for the balance years for 1 to 57 in *rāśi*, *aṁśa*, *kalā* and *vikalā*s and also *vāra*.

Example 1*:*

Śaka Year	*Rāśi*	*Aṁśa*	*Kalā*	*Vikalā*	*Vāra*
1628	8	7	42	50	0
adding	0	5	46	59	1
1685	8	13	29	49	1
adding	0	5	46	59	1
1742	8	19	16	48	2

Now, the number of days in a mean lunar month = 29.53058795, the number of days in a mean lunar year = 354.3670554 and the number of days in year having an *adhikamāsa* = 383.8976434 = 384 (approx.) since a lunar year having an *adhikamāsa* (intercalary month) will have 13 lunar months. Now, converting these days of a normal lunar year of 354 days and the lunar year with *adhikamāsa* of 384 days into *vallī* and *vāra* we obtain *vallī* = 0 | 0 | 5 | 54 *vāra* = 4 and *vallī* = 0 | 0 | 6 | 24 *vāra* = 6 respectively. We observe that these have been included in *ahargaṇavallī.* Table 8.2 for the year 1, the number of days is taken as 384, since it had an *adhikamāsa* and the corresponding *vallī* components are given as 0 | 0 | 6 | 24 and *vāra* = 6. For the next year, the number of accumulated days will be 384 + 354 = 738 and the *vallī* components corresponding to 738 days is 0 | 0 | 12 | 18 and *vāra* = 4. Similarly, for year 3 the number days is taken as 384 + 354 + 354 = 1,092. The *vallī* components are 0 | 0 | 18 | 12 and *vāra* = 0 and so on as shown in the Example 2 below.

Example 2*:*

Year	*Rāśi*	*Aṁśa*	*Kalā*	*Vikalā*	*Vāra*
1	0	0	6	24	6
adding	0	0	5	54	4
2	0	0	12	18	3
adding	0	0	5	54	4
3	0	0	18	12	0
adding	0	0	6	24	6
4	0	0	24	36	6
adding	0	0	5	54	4
5	0	0	30	30	3
adding	0	0	5	54	4
6	0	0	36	24	0

Table 8.2: Ahargaṇavallī for Years 1-57

Koṣṭaka	*Ahargaṇavallī*				
(Years)	*Rāśi*	*Aṁśa*	*Kalā*	*Vikalās*	*Vāra*
1	0	0	6	24	6
2	0	0	12	18	3
3	0	0	18	13	1
4	0	0	24	37	0
5	0	0	30	31	4
6	0	0	36	55	4
7	0	0	42	49	0
8	0	0	48	44	5
9	0	0	55	8	4
10	0	1	1	2	1
11	0	1	6	56	5
12	0	1	13	20	4
13	0	1	19	15	2
14	0	1	25	9	6
15	0	1	31	33	5
16	0	1	37	27	2
17	0	1	43	51	1
18	0	1	49	45	5
19	0	1	55	40	3
20	0	2	2	54	2
21	0	2	7	58	6
22	0	2	13	52	3
23	0	2	20	16	2
24	0	2	26	11	0
25	0	2	32	5	4
26	0	2	38	29	3
27	0	2	44	23	0
28	0	2	50	47	6
29	0	2	56	42	4
30	0	3	2	36	1
31	0	3	9	0	0
32	0	3	14	54	4
33	0	3	20	49	2

Cont.

Koṣṭaka	*Ahargaṇavallī*				
(*Years*)	*Rāśi*	*Aṁśa*	*Kalā*	*Vikalās*	*Vāra*
34	0	3	27	13	1
35	0	3	33	7	5
36	0	3	39	31	4
37	0	3	45	25	1
38	0	3	51	19	5
39	0	3	57	43	4
40	0	4	3	38	2
41	0	4	9	32	6
42	0	4	15	56	5
43	0	4	21	50	2
44	0	4	27	45	0
45	0	4	34	9	6
46	0	4	40	3	3
47	0	4	46	27	2
48	0	4	52	22	0
49	0	4	58	16	4
50	0	5	4	40	3
51	0	5	10	34	0
52	0	5	16	28	4
53	0	5	22	52	3
54	0	5	28	46	0
55	0	5	35	10	6
56	0	5	41	5	4
57	0	5	46	59	1

Table 8.3 gives *pākṣikacālanam* (fortnightly values) of *ahargaṇavallī* which is always additive.

In *pākṣikacālanam* of *ahargaṇavallī* given in Table 8.3, the last but one entry, i.e. the fourth component of *pakṣa vallī* gives the number of civil days at the end of the *pakṣa* after removing the multiples of 60.

Table 8.3: Fortnightly Values of Ahargaṇavallī

Lunar Months	*Pakṣas*	*Ahargaṇavallī*				
		Rāśi	*Aṁśa*	*Kalā*	*Vikalās*	*Vāra*
Caitra	*Śukla*	0	0	0	15	1
	Kṛṣṇa	0	0	0	30	2
Vaiśākha	*Śukla*	0	0	0	44	2
	Kṛṣṇa	0	0	0	59	3
Jyeṣṭha	*Śukla*	0	0	1	14	4
	Kṛṣṇa	0	0	1	29	5
Āṣāḍha	*Śukla*	0	0	1	43	5
	Kṛṣṇa	0	0	2	58	6
Śrāvaṇa	*Śukla*	0	0	2	13	0
	Kṛṣṇa	0	0	2	28	1
Bhādrapada	*Śukla*	0	0	2	42	1
	Kṛṣṇa	0	0	2	57	2
Āśvayuja	*Śukla*	0	0	3	12	3
	Kṛṣṇa	0	0	3	27	4
Kārttika	*Śukla*	0	0	3	41	4
	Kṛṣṇa	0	0	3	56	5
Mārgaśīrṣa	*Śukla*	0	0	4	11	6
	Kṛṣṇa	0	0	4	26	0
Puṣya	*Śukla*	0	0	4	40	0
	Kṛṣṇa	0	0	4	55	1
Māgha	*Śukla*	0	0	5	10	2
	Kṛṣṇa	0	0	5	25	3
Phālguna	*Śukla*	0	0	5	40	4
	Kṛṣṇa	0	0	5	54	4

Example: At the end of 7 *pakṣas*, the number of civil days

$$= \frac{\text{Duration of a lunar month}}{2} \times 7 = \frac{29.53058795}{2} \times 7$$

$$= 103.3570578 = 43.357057 = 43$$

by removing multiples of 60 and taking the integer value.

PROCEDURE FOR FINDING AHARGAṆAVALLĪ FROM THE MAKARANDASĀRIṆĪ TABLES

The working procedure for finding *ahargaṇavallī* using the *Makarandasāriṇī* tables is as given below:

i. Subtract the nearest Śaka year given in Table 8.1 from *iṣṭa* Śaka (given Śaka year, for which *aharganaṇavallī* is to be found) and obtain the difference called *śeṣa* (remainder).

ii. Find the *vallī* values corresponding to the nearest Śaka year given in the table and also for the *śeṣa varṣa* (remainder) using Tables 8.1 and 8.2 respectively. Also find the *vāra* corresponding to these given in the last columns of Tables 8.1 and 8.2.

iii. Add the *vallī* and *vāra* for the Śaka year and the remainder correspondingly. Remove the multiples of 7 from *vāra* (when it exceeds 7).

iv. The above sum gives *grahadinavallī* or *ahargaṇavallī* for the *iṣṭa* Śaka year (given Śaka year).

v. Now, using Table 8.3, obtain the *pakṣavallī* and *vāra* for the given *pakṣa* of the running lunar month of the given Śaka year.

vi. Add the *pakṣavallī* and *vāra* obtained in the above step (v) to the *grahadinavallī* or *ahargaṇavallī* of *iṣṭa* Śaka year obtained in step (iv).

vii. Add the elapsed number of *tithi*s of the running *pakṣa* of the lunar month to the sum obtained above in step (vi) in the fourth component of the *vallī.* This gives the *ahargaṇavallī* or *ahargaṇadinavallī* for the given day of the Śaka year.

Example: Given date: Śaka 1534 Vaiśākha *śukla* 15 corresponding to 15 May 1612.

The nearest Śaka year from Table 8.1 is 1514.

Given Śaka year – Nearest Śaka year from Table 8.1 = 1,534 – 1,514
= 20 (*śeṣa*)

Now, *vallī* corresponding to 1514 is → 7 | 56 | 08 | 52 and *vāra* 5

Vallī corresponding to *śeṣa varṣa*, 20 is → 0 | 02 | 02 | 04 and *vāra* 2

Pakṣavallī for Vaiśākha *śukla* 15 is → 0 | 00 | 00 | 44 and *vāra* 2

Adding → 7 | 58 | 11 | 40 and *vāra* 2.

Thus, *ahargaṇavallī* for the given day is 7 | 58 | 11 | 40 and *vāra* 2 (removing multiples of 7).

Note: (i) *vāra* is counted from Sunday as 0.

(ii) While adding the *vallī* components, multiples of 60 are removed.

Example: The given date: Śaka 1891 Śrāvaṇa *kṛṣṇa pratipatā* corresponding to 31 July 1969.

The nearest Śaka year from Table 8.1 is 1856.

Given Śaka year – Nearest Śaka year from Table 8.1:

1891 – 1856 = 35 (*śeṣa*)

Now, *vallī* corresponding to 1856 is → 8 | 30 | 50 | 46 and *vāra* 4

Vallī corresponding to *śeṣa varṣa*, 35 is → 0 | 03 | 33 | 07 and *vāra* 1

Pakṣavallī for *Śrāvaṅa Kṛṣṇ*a *pratipatā* is → 0 | 00 | 02 | 15 and *vāra* 1

Adding → 8 | 34 | 26 | 08 and *vāra* 6

Thus, *aharganavallī* for the given day is 8 | 34 | 26 | 08 and *vāra* 6 (removing multiples of 7).

Example: Given date: Śaka 1939 Vaiśākha *śukla* 15 corresponding to 10 May 2017, Wednesday

The nearest Śaka year from Table 8.1 is 1913.

Given Śaka year – Nearest Śaka year from Table 8.1:

1939 – 1913 = 26 (*śeṣa*)

Now, *vallī* corresponding to 1913 is → 8 | 36 | 37 | 45 and *vāra* 5

Vallī corresponding to *śeṣa varṣa*, 26 is → 0 | 02 | 38 | 29 and *vāra* 3

Pakṣavallī for Vaiśākha *śukla* 15 is → 0 | 00 | 00 | 44 and *vāra* 2

Adding → 8 | 39 | 16 | 58 and *vāra* 3

Thus, *aharganavallī* for the given day is 8 | 39 | 16 | 58 and *vāra* 3 (removing multiples of 7).

Example: Given date: Śaka 1939 Śrāvaṇa *kṛṣṇa saptamī* (7) corresponding to 14 August 2017, Monday.

Nearest Śaka year from Table 8.1 is 1913.

Given Śaka year – Nearest Śaka year from Table 8.1 = 1939 – 1913

= 26 (*śeṣa*)

Now, *vallī* corresponding to 1913 is → 8 | 36 | 37 | 45 and *vāra* 5

Vallī corresponding to *śeṣa varṣa*, 26 is → 0 | 02 | 38 | 29 and *vāra* 3

Pakṣavallī for Śrāvaṇa *śukla* is → 0 | 00 | 02 | 13 and *vāra* 0

number of *tithi*s in the given *pakṣa* is → 0 | 00 | 00 | 07 and *vāra* 0

Adding → 8 | 39 | 18 | 34 and *vāra* 1

Thus, *ahargaṇavallī* for the given day is 8 | 39 | 18 | 34 and *vāra* 1 (removing multiples of 7).

OBTAINING AHARGAṆAVALLĪ FROM KALI AHARGAṆA DAYS

Let A be the Kali *ahargaṇa* for a given date.

1. Divide A by 60, consider the integer part of the quotient Q_1 and remainder R_1.
2. Divide Q_1 by 60, consider the quotient Q_2 and the remainder R_2.
3. Divide Q_2 by 60 and consider the quotient Q_3 and remainder R_3.
4. Divide Q_3 by 60 to get quotient Q_4 and remainder R_4.

Now *ahargaṇavallī* for the given date is given by $R_4 \mid R_3 \mid R_2 \mid R_1$.

Example: Śaka 1849, Mārgaśira *śukla* 15 Thursday corresponding to 8 December 1927.

The Kali *ahargaṇa* for the given date, A = 1,836,758.

Now, dividing A by 60, integer part of the quotient Q_1 = 30,612 and remainder $R_1 = 38$

dividing Q_1 by 60, integer part of the quotient Q_2 = 510 and remainder $R_2 = 12$

dividing Q_2 by 60, integer part of the quotient Q_3 = 8 and remainder $R_3 = 30$

dividing Q_3 by 60, integer part of the quotient Q_4 = 0 and remainder $R_4 = 8$

Ahargaṇavallī for the given date is $R_4 \mid R_3 \mid R_2 \mid R_1$= 8 | 30 | 12 | 38.

OBTAINING KALI AHARGAṆA DAYS FROM AHARGAṆAVALLĪ

Ahargaṇavallī for the given date is of the form $V_1 \mid V_2 \mid V_3 \mid V_4$ which is equal to the remainders $R_4 \mid R_3 \mid R_2 \mid R_1$, i.e. $V_1 = R_4$, $V_2 = R_3$, $V_3 = R_2$, $V_4 = R_1$.

Above steps 1, 2, 3 and 4 give the equations.

$A = Q_1 \times 60 + R_1$, $Q_1 = Q_2 \times 60 + R_2$, $Q_2 = Q_3 \times 60 + R_3$, and $Q_3 = Q_4 \times 60 + R_4$.

From the above equations we obtain

$$\begin{aligned}
A &= (Q_1 \times 60) + R_1 \\
&= ((Q_2 \times 60) + R_2) \times 60 + R_1 \\
&= (Q^2 \times 60^2) + (R_2 \times 60) + R_1 \\
&= ((Q_3 \times 60) + R_3) \times 60^2 + (R_2 \times 60) + R_1 \\
&= Q_3 \times 60^3 + (R_3 \times 60^2) + (R_2 \times 60) + R_1 \\
&= ((Q_4 \times 60) + R_4) \times 60^3 + (R_3 \times 60^2) + (R_2 \times 60) + R_1 \\
&= (Q_4 \times 60^4) + (R_4 \times 60^3) + (R_3 \times 60^2) + (R_2 \times 60) + R_1
\end{aligned}$$

$$A = (Q_4 \times 60^4) + (R_4 \times 60^3)+(R_3 \times 60^2) + (R_2 \times 60) + R_1. \quad (1)$$

In the process of finding *vallī* from *ahargaṇa* we repeat the division by 60 till we get the quotient 0 and remainder less than 60. In the 4th stage we obtain $Q_4 = 0$ and $R_4 < 60$. In view of this equation (1) becomes

$$A = (R_4 \times 60^3) + (R_3 \times 60^2) + (R_2 \times 60) + R_1,$$

$$\text{i.e. } A = (V_1 \times 60^3) + (V_2 \times 60^2) + (V_3 \times 60) + V_4, \quad (2)$$

where V_1, V_2, V_3, V_4 are components of *vallī* (given in the *Makarandasariṇī* Tables 8.1-8.3).

Example: For the given date: Śaka 1891 Śrāvaṇa *kṛṣṇa pratipatā* corresponding to 3 July 1969, Thursday.

dinavallī for the given date = 8 | 34 | 26 | 8, i.e.

$$V_1 = 8,\ V_2 = 34,\ V_3 = 26,\ V_4 = 8.$$

ahargaṇa, $A = (V_1 \times 60^3) + (V_2 \times 60^2) + (V_3 \times 60) + V_4$

$$= 8 \times 60^3 + 34 \times 60^2 + 26 \times 60 + 8$$

$$= 1{,}728{,}000 + 122{,}400 + 1{,}560 + 8 = 1{,}851{,}968.$$

Using modern tables, *ahargaṇa* for 31 July 1969 Thursday is 1,851,968.

Example : Given date: Śaka 1534 Vaiśākha *śukla* 15 corresponding to 15 May 1612 Tuesday.

ahargaṇavallī = 7 | 58 | 11 | 40, i.e. $V_1 = 7, V_2 = 58, V_3 = 11, V_4 = 40$.

$\therefore$ *ahargaṇa*, $A = (V_1 \times 60^3) + (V_2 \times 60^2) + (V_3 \times 60) + V_4$

$$= 7 \times 60^3 + 58 \times 60^2 + 11 \times 60 + 40 = 1{,}721{,}500.$$

Using modern tables, *ahargaṇa* for 15 May 1612 is 1,721,500.

Example: Given date: Śaka 1939 Vaiśākha *śukla* 15 corresponding to 10 May 2017 Wednesday.

ahargaṇavallī = 8 | 39 | 16 | 58, i.e. $V_1 = 8, V_2 = 39, V_3 = 16, V_4 = 58$.

$\therefore$ *ahargaṇa*, $A = (V_1 \times 60^3) + (V_2 \times 60^2) + (V_3 \times 60) + V_4$

$$= 8 \times 60^3 + 39 \times 60^2 + 16 \times 60 + 58 = 1{,}869{,}418.$$

Using modern tables *ahargaṇa* for 10 May 2017 is 1,869,418.

Example: Given date: Śaka 1939 Śrāvaṇa *kṛṣṇa saptamī* (7) corresponding to 14 August 2017 Monday.

ahargaṇavallī = 8 | 39 | 18 | 34, i.e. $V_1 = 8, V_2 = 39, V_3 = 18, V_4 = 34$.

ahargaṇa, $A = (V_1 \times 60^3) + (V_2 \times 60^2) + (V_3 \times 60) + V_4$

$$= 8 \times 60^3 + 39 \times 60^2 + 18 \times 60 + 34 = 1{,}869{,}514.$$

Finding True Positions of Five Star Planets

In finding the true position of star planets we need to apply two major corrections, viz. *mandaphala* (the equation of centre) and *śīghraphala* (the equation of conjunction). According to the *Makarandasariṇī*, *mandaphala* of the five star planets differ from those of the other texts. In the *Sūryasiddānta*, the true position of star planet is obtained by applying successively four corrections. Among these, *manda* correction is applied twice in between two *śīghra* corrections. On the other hand, the *Makarandasariṇī* simplifies the procedure by reducing it to three corrections. Here *manda* correction is applied only once between two *śīghra* corrections. In the process, *mandakarṇa* has consolidated the two *manda*s of the *Sūryasiddānta* into a single equation in the *Makarandasariṇī*. This makes the *mandaphala* value of the *Makarandasariṇī* differ from those of other texts.

Mean sun is taken as *śīghrocca* for Kuja, Guru and Śani. For Budha and Śukra, *budhocca* and Śukra *śīghrocca* are obtained as explained in the text. Mean sun is also considered as Mean Budha and Mean Śukra.

Śīghrakendra = mean planet – *Śīghrocca*

If *śīghrakendra* is greater than 6^R, then subtract it by 12^R.

PROCEDURE FOR FINDING ŚĪGHRAPHALA USING THE MAKARANDASARIṆĪ TABLES

1. Find *śīghrakendra* of the planet.
2. Find the values (in degrees, etc.) in the column headed by the number in *aṁśa* (degree) place of *śīghrakendra*, using *śīghra-mandaphala* table of the corresponding planet. This is called as first *śīghrāṅka*.
3. Find the values from the next column (*agrimāṅka*). This is called as second *śīghrāṅka*.
4. Consider the difference (3) – (2) and multiply this difference by the *kalā*, *vikalā* of *śīghrakendra* and divide by 60 to obtain the result in *kalā*s.
5. The above result obtained in (4) is added to or subtracted from first *śīghrāṅka* obtained in step (1). This is *śīghraphala* in degree, etc.
6. *Śīghraphala* is additive if *śīghrakendra* is < 180˚ and is subtractive if *śīghrakendra* is > 180˚.

FINDING MANDAPHALA USING TABLES

1. Find *mandakendra* of planet where

   Mandakendra = mean planet – *mandocca*
2. If *mandakendra* > 6^R, then subtract it from 12^R and consider degrees, etc.
3. Find the entries in the column headed by the number present in degree position of *mandakendra*. This is called first *mandāṅka*. Also find the entries in the next column to this number (*agrimāṅka*) using *mandaphala* table. This is called second *mandāṅka*.

4. Consider the difference between first and second *mandāṅka*s multiply this difference by *kalā*, *vikalā*, etc. of *mandakendra* and divide by 60 to obtain the result in *kalā*s.
5. Add this to or subtract *kalā* from first *mandāṅka* obtained against degree number of *mandakendra*. The result is called *mandaphala*.

APPLICATION OF ŚĪGHRA AND MANDA CORRECTIONS

1. Obtain *śīghraphala* as explained earlier, consider half of it (*ardhaśīghraphala*).
2. This half *śīghraphala* is added to or subtracted from mean planet to get half *śīghra* corrected mean planet.
3. Find *mandaphala* and add (or subtract) *mandaphala* as it is to mean planet to get *manda* corrected planet.
4. Considering *manda* corrected planet as mean planet, obtain second *śīghraphala* and correct the *manda* corrected planet with *śīghra* correction. This gives true position of the planet. I.e.

   if $MP \equiv$ mean planet
   $P_1 = ½$ *śīghra* corrected planet
   $P_2 =$ *manda* corrected planet
   $P_3 =$ true planet,

   then

   $P_1 = MP + ½$ *śīghraphala*
   $P_2 = MP +$ *mandaphala*
   $P_3 = P_2 +$ *śīghraphala*.

FINDING TRUE DAILY MOTIONS OF PLANETS

1. Consider the *mandāṅka* difference obtained in finding *mandaphala*.
2. Multiply this *mandāṅkāntara* by mean daily motion of the planet. The result will be in *kalā*s.
3. Add this to or subtract this from (according as *makarādi ṛṇam* and *karkādi dhanam*) mean daily motion to get *manda* corrected motion.
4. *Śīghrakendra gati* = *ṣīghrocca gati* – *manda* corrected motion.

5. Consider the difference between *śīghrāṅka*s obtained during second *śīghra* correction (*antimaśīghraphalam*) and multiply this difference by *śīghrakendragati*. Divide the product by 60.
6. The above result is added to or subtracted from *manda* corrected motion to obtain corrected daily motion of the planet.
7. Do the reverse for retrograde motion.

Note: In the present paper the tables for the *manda* and *śīghra* corrections are given for Kuja and Budha. The tables for other bodies are not given since these tables occupy a lot of space.

Kuja's Mandaphala and Śīghraphala Table

91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
33	33	33	33	34	34	34	35	35	35	35	36	36	36	36	37	37	37	37	37	38	38	38	38	38	38	39	39	39	39
6	23	40	57	13	29	46	2	34	34	49	4	18	33	47	0	13	26	39	52	4	15	26	37	47	57	7	16	25	33
11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	10	10	10	10	10	10	10
30	31	32	32	32	32	32	32	31	31	30	29	27	25	23	21	19	16	13	10	7	4	1	57	52	48	44	39	34	29
121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144	145	146	147	148	149	150
39	39	39	39	40	40	40	40	40	40	40	40	40	40	40	40	39	39	39	39	39	39	38	38	38	38	37	37	36	36
40	46	52	58	3	7	10	13	15	16	16	16	15	12	8	3	57	50	42	33	22	9	55	40	22	5	45	22	57	31
10	10	10	10	10	10	9	9	9	9	9	9	9	8	8	8	8	8	8	8	7	7	7	7	7	7	6	6	6	6
24	18	13	7	1	54	47	40	33	26	19	11	3	55	47	39	30	21	12	3	54	44	34	24	14	3	53	43	32	21
151	152	153	154	155	156	157	158	159	160	161	162	163	164	1565	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180
36	35	34	34	33	33	32	31	30	29	28	27	26	25	24	23	21	20	18	17	15	14	12	10	9	7	5	3	1	0
2	30	57	20	42	0	16	28	38	44	47	46	42	34	22	7	49	26	59	29	57	20	41	58	13	25	36	45	53	0
6	5	5	5	5	5	5	4	4	4	4	4	3	3	3	3	2	2	2	2	2	1	1	1	1	0	0	0	0	0
10	59	48	36	25	13	1	49	37	25	13	0	47	35	22	9	56	42	29	16	3	49	36	22	9	56	42	28	14	0

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	0	1	1	1	2	2	3	3	3	4	4	5	5	5	6	6	7	7	7	8	8	9	9	9	10	10	10	11	11
24	47	11	35	58	22	46	9	33	56	20	44	7	30	55	19	41	5	28	51	15	69	3	26	49	12	35	58	21	44
0	0	0	0	0	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	4	5	5	5
12	23	34	45	46	7	18	29	40	51	2	13	24	35	45	56	7	18	28	39	49	59	10	20	30	40	51	1	11	20
31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
12	12	12	13	13	14	14	14	15	15	15	16	16	17	17	17	18	18	18	19	19	20	20	20	21	21	21	22	22	22
7	30	53	16	39	1	24	47	10	33	55	18	41	3	25	48	10	32	54	17	39	1	23	45	7	29	50	12	34	56
5	5	5	5	6	6	6	6	6	6	7	7	7	7	7	7	7	8	8	8	8	8	8	8	8	9	9	9	9	9
30	40	50	59	9	18	27	36	45	54	3	12	20	29	38	46	54	2	10	18	26	34	42	49	56	3	10	17	23	30
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
33	23	23	24	24	25	25	25	26	26	26	27	27	27	28	28	28	29	29	29	30	30	30	30	31	31	31	32	32	32
16	37	59	20	41	2	23	44	4	24	45	5	25	45	5	25	45	5	24	43	3	29	40	59	17	36	54	12	30	48
9	9	9	9	10	10	10	10	10	10	10	10	10	10	10	10	10	11	11	11	11	11	11	11	11	11	11	11	11	11
36	43	49	55	1	7	12	17	22	27	32	36	41	45	49	53	57	0	4	7	10	13	16	18	20	22	24	26	27	29

Example: Finding True Kuja

Mean Kuja = $9^R29°21'13''$; Mean Sun = $1^R4°57'24''$ (mean positions are taken from Viśvanātha's example for the date Śaka 1534 Vaiśākha *śukla* 15 corresponding to 15 May 1612).

Śīghrakendra = mean Kuja – mean Ravi (*śīghrocca*)

= 9 | 29 | 21 | 13 – 1 | 4 | 57 | 24

= 8 | 24 | 23 | 49

Since *śīghrakendra* > 6^R, subtracting it from 12^R, we get

Śīghrakendra = $12^R - 8^R$ | 24° | 23' | 49'' = $3^R5°36'11''$ = 95°36'11''

Bhujāṁśa of *śīghrakendra* = 95°36'11''

Number degree position = 95 remaining *kalādi* = 36'.

From the table of *manda* and *śīghraphala* for Kuja the entry in the column headed by 95, first *śīghrāṅka* = 34 | 13 second *śīghrāṅka* = 34 | 29

difference (first – second) = 34 | 13 – 34 | 29 = – 0 | 16.

Now

$$\frac{\text{Difference between } \textit{Śīghrāṅka} \times \textit{kalādi} \text{ of } \textit{bhujāṁśa}}{60}$$

$$= \frac{0|16 \times 36|11}{60} = \frac{-9|38|56}{60} = 0|9|38.93 \approx -0|9|39$$

Now, first *śīghrāṅka* = (– 0 | 9 | 39) = 34 | 13 + 0 | 9 | 39 = 34 | 22,

i.e. *śīghraphala* = 34 | 22 | 39.

P_1 = ½ *śīghra* corrected planet

= mean planet – ½ *śīghraphala*

= $9^R29°21'13''$ + (34 | 22 | 39)/2

P_1 = $10^R16°32'32''$.

Mandocca of Kuja = 4^R | 10°

Mandakendra = P_1, ½ *śīgh*. corr. Kuja – *Mandocca*

= $10^R16°32'32''$ – 4^R | 10° = $6^R\,6°32'32''.5$.

Since *mandakandra* > 6^R, *bhujāṁśa* of *mandakandra* = $12^R - 6^R\,6°32'32''.5$

= 173°27'27''.5

*Mandāṅka*s in the columns headed by 173 and 174; first *mandāṅka* = 1 | 36, second *mandāṅka* = 1 | 22. Difference = 1 | 36 – 1 | 22 = 0 | 14.

Now,

$$\frac{\text{Difference between } \textit{mandākas} \times \textit{kalādi} \text{ of } \textit{bhujāṁśa}}{60}$$

$$= \frac{0|14 \times 27'27''.5}{60} = 0|6|24.42.$$

Mandaphala = first *mandāṅka* – 0 | 6 | 24.42

= 1 | 36 – 0 | 6 | 24.42 = 1 | 29 | 36 (+ve)

P_2, *manda* corrected Kuja = mean planet + m*andaphala*

= $9^R 29°21'13'' + 1° | 29' | 36''$

= $10^R 00°50'49'' = 10^R 0°50'49''$

Second *śīghrakendra* = P_2,

manda corrected planet – *śīghraocca*

= $10^R 0°50'49'' - 1^R | 4° | 57' | 24''$

= $265°53'25'' = 8^R 25°53'25''$

Bhujāṁśa of *śīghrakendra* = $12^R - 8^R 25°53'25''$

= $94°6'35''$

first *śīghrāṅka* = 33 | 57 (corresponding to 94)

second *śīghrāṅka* = 34 | 13 (corresponding to 95)

$$\frac{\text{Difference} \times \textit{kalādi}}{60} = \frac{(33|57 - 34|13) \times 6'35''}{60} = -\,0|4|45.33.$$

Śīghraphala = First *śīghrāṅka* – (– 0 | 1 | 45.33)

= 33 | 57 + 0 | 1 | 45.33 = 33 | 58 | 45.33

Śīghra corrected Kuja = *manda* corrected *Kuja* + *śīghraphala*

= $10^R 0°50'49'' + 33°58'45''.33$

i.e. true Kuja = $11^R 4°49'34''$

To find True daily motion of Kuja:

Difference between *mandāṅka*s = 0 | 14 = 14

Mean motion of Kuja = 31 | 26 |

Now,

$$\frac{14' \times 31'|26}{60} = \frac{440|4}{60} = 7|20|4\,(+\text{ve})$$

Manda corrected daily motion = mean motion ± 7 | 20 | 4

= 31 | 26 + 7 | 20 = 38 | 46

Śīghrocca gati = daily motion of Ravi = 59 | 8.

Now,

Śīghrakendra gati = *śīghrocca gati* – *manda* corrected *gati*

= 59 | 8 – 38 | 46

= 20 | 22

Difference between *śīghrāṅka*s of second *śīghra* correction

= 33 | 57 – 34 | 13 = – 0 | 16 = 16.

Now,

$$\frac{(\textit{Śīghrakendra gati} \times 16)}{60} = \frac{325|52}{60} = 5|25.$$

True daily motion of Kuja = *manda* corrected motion + 5 | 25

= 38 | 46 + 5 | 25

= 44' | 11".

Example 2: Finding true position of Budha and true daily motion of Budha (mean Budha = mean Ravi)

Śīghrakendra of Budha = mean Budha – *śīghrocca* of Budha

= mean Sun – *budhocca*

= $1^R 4°57'24'' – 3^R 1°44'33''$

= $– 56°47'9'' + 12^R$

= $10^R\ 3°12' | 51'' |$

Bhujāṁśa = $12^R – 10^R 3°12' | 51'' |$ = $56°47' | 9'' |$

First *śīghrāṅka* corresponding to 56 = 14 | 10

Second *śīghrāṅka* corresponding to 57 = 14 | 23

$$\textit{Śīghraphala} = \text{first } \textit{śīghrāṅka} - \frac{\text{difference of } \textit{śīghrāṅka} \times \textit{kalā}}{60}$$

Budha's *Mandaphala* and *Śīghraphala* Table

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
0	0	0	1	1	1	1	2	2	2	2	3	3	3	4	4	4	4	5	5	5	5	6	6	6	6	7	7	7	7
17	33	49	5	21	37	53	9	26	42	58	14	30	46	2	18	34	50	6	21	37	53	9	24	39	55	10	25	41	57
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2
5	10	14	19	24	28	33	38	43	47	52	56	1	5	10	14	19	24	28	32	36	41	45	49	54	58	2	6	10	14
31	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
8	8	8	8	9	9	9	9	10	10	10	10	11	11	11	11	12	12	12	12	13	13	13	13	13	14	14	14	14	15
12	27	42	57	12	27	43	57	12	27	42	56	11	25	40	54	8	22	36	49	3	17	31	44	57	10	23	36	49	2
2	2	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
18	22	25	29	33	36	40	44	47	51	55	59	2	5	8	11	14	17	20	23	26	29	32	34	37	40	42	45	47	50
61	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**	**81**	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**
15	15	15	15	16	16	16	16	16	17	17	17	17	17	17	18	18	18	18	18	18	19	19	19	19	19	19	19	20	20
14	27	39	51	3	15	27	39	50	2	13	24	34	44	55	6	16	26	36	45	54	3	11	20	29	37	45	53	0	7
3	3	3	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
53	54	56	58	0	2	4	6	8	9	10	11	12	13	15	17	18	19	20	22	23	24	25	25	25	26	26	27	227	27

91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
20	20	20	20	20	20	20	20	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21
14	21	28	35	41	46	51	56	1	5	10	14	18	21	23	25	27	29	31	31	31	31	31	30	29	28	25	22	19	16
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3
27	28	28	27	27	27	27	26	26	25	24	24	23	22	21	20	19	18	17	16	14	12	11	9	8	6	4	2	0	58
121	**122**	**123**	**124**	**125**	**126**	**127**	**128**	**129**	**130**	**131**	**132**	**133**	**134**	**135**	**136**	**137**	**138**	**139**	**140**	**141**	**142**	**143**	**144**	**145**	**146**	**147**	**148**	**149**	**150**
21	21	21	20	2020	20	20	20	20	20	20	19	19	19	19	19	18	18	18	18	17	17	17	17	16	16	16	15	15	15
12	8	3	57	51	44	36	29	21	13	4	54	43	32	20	8	55	42	27	13	59	42	24	6	48	29	9	49	28	6
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
56	54	52	49	47	44	41	39	36	33	30	27	24	21	18	15	12	8	5	2	59	55	51	47	43	39	35	31	27	23
151	**152**	**153**	**154**	**155**	**156**	**157**	**158**	**159**	**160**	**161**	**162**	**163**	**164**	**165**	**166**	**167**	**168**	**169**	**170**	**171**	**172**	**173**	**174**	**175**	**176**	**177**	**178**	**179**	**180**
14	14	13	13	13	12	12	11	11	10	10	9	9	8	8	7	7	6	6	5	5	5	4	3	2	2	1	1	0	0
44	21	58	34	9	43	17	49	23	55	27	57	27	57	56	55	23	50	18	45	12	38	4	29	55	20	45	11	36	0
2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
19	15	11	6	2	57	53	48	44	39	35	30	25	20	16	11	6	1	56	51	46	40	36	31	26	21	15	10	5	0

$$= 14 \mid 10 - \left[\frac{(14|10 - 14|23) \times 47'9''}{60} \right]$$

$$= 14 \mid 10 - (-\,0 \mid 10 \mid 12.95)$$

$$= 14 \mid 20 \mid 12.95.$$

P_1 = Half *śīghra* corrected Budha = mean Budha + ½ *śīghraphala*

$$= 1^R 4^\circ 57' \mid 24'' \mid \; + 14 \mid 20 \mid 12.95$$

$$= 34^\circ 57' \mid 24'' \mid \; + 7 \mid 10 \mid 6.48$$

$$= 42^\circ 7' \mid 30'' \mid \; = 1^R 12^\circ 7' \mid 30'' \mid$$

P_1 = Half *śīghra* corrected Budha = $1^R 10°7' | 30'' |$.

Mandocca of Budha = 7 | 10.

Mandakendra = Half *śīghra* corrected Budha – *mandocca*

$= 1^R 12°7' | 30'' | - 7^R 10°$

$= 182°7' | 30'' | = 6^R 2°7' | 30'' |$

Bhujāṁśa of *mandakendra* = $360° - 182°7' | 30'' | = 177°52' | 30'' |$

Mandaphala

$$= \text{First } \textit{Mandāṅka} - \frac{\begin{pmatrix}\text{Differrence in}\\ \textit{mandāṅka}\text{s}\end{pmatrix} \times \begin{pmatrix}\textit{kalādi} \text{ of}\\ \textit{bhujāṁśa}\end{pmatrix}}{60}$$

$$= 0|15 - \frac{(0|15 - 0|10) \times 52'30''}{60} = 0|15 - 0|4|22$$

$= 0 | 10 | 37$ (– ve)

Manda corrected Budha = mean Budha (mean Sun) – *mandaphala*

$= 1^R 4°57' | 24'' | - (-0 | 10 | 37)$

$= 1^R 4°57' | 24'' | + 0°10' | 37'' |$

$= 1^R 5°8' | 1' | = 35°8' | 1'' |$

Second *śīghrakendra* = *manda* corrected Budha – *śīghrocca* of Budha

$= 1^R 5°8' | 1'' | - 3^R\, 1°44' | 33'' |$

$= 303°23' | 28'' | = 10^R 3°23' | 28'' |$

Bhujāṁśa of *SK* = $12^R - 10^R 3°23' | 28'' | = 56°36' | 32'' |$

Śīghraphala

$$= \text{First } \textit{śīghrāṅka} - \frac{\begin{pmatrix}\text{Difference between}\\ \textit{śīgrāṅka}\text{s}\end{pmatrix} \times \begin{pmatrix}\textit{kalādi} \text{ of}\\ \textit{bhujāṁśa}\end{pmatrix}}{60}$$

$$= 14 | 10 - \left[\frac{(14|10 - 14|23) \times 36'32''}{60}\right]$$

$= 14 | 10 - (-0 | 7 | 54.93)$

$= 14 | 17 | 54.93$

True Budha = *manda* corrected Budha – *śīghraphala*

$= 1^R 5°8' | 1'' | - 14°17' | 54'' | .93.$

True Budha = $1^R 19°25' | 55'' | .93 = 49°25' | 55'' | .93.$

To find true daily motion of Budha

Difference between *mandāṅka*s = (0 | 15 – 0 | 10) = 0 | 5

Mean motion of Budha = mean motion of Sun = 59 | 8

(Mean motion × difference between *mandāṅkās*)

= 59 | 8 × 0 | 5

= 4 | 55 | 40 (–ve).

Manda corrected motion = 59 | 8 – 4 | 55 = 54 | 13.

Mean daily motion of *śīghrocca* of Budha = 245 | 32.

Śīghrakendra gati = *śīghrocca gati* – *manda* corrected *gati*

= 245 | 32 – 54 | 18 = 191 | 19.

Śīghragatiphalam

= (*śīghrakendragati*) × (difference between second *śīghrāṅka*s)

= (191 | 19) × (14 | 10 – 14 | 23)

= – 41 | 27.

Śīghra corrected motion = *manda* corrected motion – (– 41 | 27)

= 54 | 13 + (41 | 27)

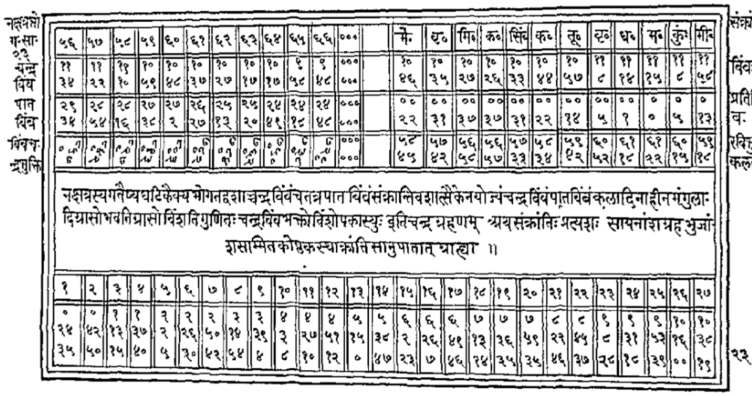
= 95 | | 40 | ,

i.e. true daily motion of Budha = 95'40".

Angular Diameter

Diameters of the sun, the moon and the earth's shadow are important in the computation of lunar and solar eclipses. In astronomy, the sizes of objects in sky are often given in terms of their angular diameters as seen from the earth. The angular diameter of an object is the angle the object subtends as seen by the observer on the earth. These angular diameters play an important role in the procedures of computation of lunar and solar eclipses, conjunction, occultation and transits.

In Indian classical astronomical texts, the procedures for calculating angular diameters (*bimba*s) are given in different forms in different texts. Majority of the Siddhāntic texts give *bimba*s in terms of the true daily motions of the sun and the moon. But some other texts including astronomical tables like the *Makarandasāriṇī*, determine *bimba*s as a function of running *nakṣatrabhoga* or *manda* anomalies of the sun and the moon.



नक्षत्रभो ग.सा.	५६	५७	५८	५९	६०	६१	६२	६३	६४	६५	६६	०००		मे.	वृ.	मि.	क.	सिं.	क.	तू.	वृ.	ध.	म.	कुं.	मी.	संक्रांतयः
चन्द्र विंब	११ ३४	११ २२	११ १०	१० ५९	१० ४८	१० ३७	१० २७	१० १७	१० १७	९ ५८	९ ४८	००० ०००		१० ४६	१० ३५	१० २७	१० २६	१० ३३	१० ४४	१० ५७	११ ८	११ १४	११ २५	११ ८	११ ५८	विंवः
पात विंब	२९ ३४	२८ ५४	२८ १६	२७ ३८	२७ २	२६ २७	२५ १३	२५ २०	२४ ४९	२४ १८	२४ ४८	००० ०००		०० २२	०० ३१	०० ३७	०० २७	०० ३१	०० २२	०० १४	०० ५	०० १	० ०	० ५	० १३	प्रतिविंवः
विंवभ. क्तभुक्ति	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	००० ०००		५८ ४५	५७ ४२	५६ ५८	५६ ५७	५७ ३३	५८ ३४	५९ ४२	६० ५२	६१ १८	६१ २२	६० १५	५९ १८	रविभुक्ति कलादि

नक्षत्रस्यगतैष्यघटिकैक्यभोगतद्वशाच्चन्द्रविंवंचतत्रपात विंवंसंक्रान्तिवशात्सैकेनयोज्यंचन्द्रविंवंपातविंबंकलादिनाहीनमंगुलादिग्रासोभवतिग्रासो विंशतिगुणितः चन्द्रविंवभक्तोविंशोपकास्युः इतिचन्द्रग्रहणम् अथसंक्रांतिः अत्यशः सायनांशग्रहभुजांशसाम्मितकोष्ठकस्थाक्रांतिसानुपातात्ग्राह्या ॥

१	२	३	४	५	६	७	८	९	१०	११	१२	१३	१४	१५	१६	१७	१८	१९	२०	२१	२२	२३	२४	२५	२६	२७
० ३४ ३५	० ४२ ५०	१ १३ १५	१ ३७ ४०	२ २ ५	२ २६ ३०	२ ५० ४२	३ १४ ५४	३ ३९ ४	४ ३ ८	४ २७ १०	४ ५१ १२	५ १५ ०	५ ३८ ४७	६ २ २३	६ २६ ७	६ ४९ ४६	७ १३ २४	७ ३६ ३५	७ ५९ ३५	८ २३ ४६	८ ४५ ३७	९ ८ २८	९ ३१ १८	९ ५३ २९	१० १६ ००	१० ३८ ३१

२२

Image of the Table for Angular Diameters

The diameters are expressed in different units in different texts. The famous classical Siddhāntic text the *Sūryasiddhānta* gives diameters in terms of linear unit *yojana*. In Brahmagupta's *Khaṇḍakhādyaka* and the *vākya* system, the angular diameters are given in minutes of arc (*kalā*s), whereas in the *Karaṇakutūhala* of Bhāskara II and the *Grahalāghava* of Gaṇeśa Daivajña, the unit used for diameter is *aṅgula*.

OBTAINING DIAMETERS OF THE SUN, THE MOON AND THE EARTH'S SHADOW-CONE ACCORDING TO THE MAKARANDASARIṆĪ

The following tables (Tables 8.4-8.5) are given in the *Makarandasariṇī* for computing angular diameters of the sun, the moon and earth's shadow-cone. In Table 8.4, the angular diameter of the moon

Table 8.4: Table for Finding the Moon's Diameter

Nakṣatrabhoga (in *ghaṭi*s)	56	57	58	59	60	61	62	63	64	65	66
Candra bimba (in *aṅgula*s)	11 34	11 22	11 10	10 59	10 48	10 37	10 27	10 17	10 17	9 58	9 48
*Pāta bimba*s (in *aṅgula*s)	29 34	28 54	28 16	27 38	27 2	26 27	25 13	25 20	24 49	24 18	24 48
Bimba candra bhukti (in *kalā*s)	857	842	827	813	800	787	774	762	750	738	727

Table 8.5: Table for Finding the Sun's Diameter

Saṅkrānti	*Ravi Bimba* (*in Aṅgulas*)		*Pāta Bimbas* (*in Aṅgulas*)		*Ravi Bhukti* (*in Kalās*)	
Meṣa	10	46	0	22	58	45
Vṛṣabha	10	35	0	31	57	42
Mithuna	10	27	0	37	56	58
Karkātaka	10	26	0	37	56	57
Siṁha	10	33	0	31	57	33
Kanyā	10	44	0	22	58	34
Tulā	10	57	0	14	59	42
Vṛścika	11	8	0	5	60	52
Dhanuṣ	11	14	0	1	61	18
Makara	11	15	0	0	61	22
Kumbha	11	8	0	5	60	15
Mīna	10	58	0	13	59	18

(*candra bimba*) and earth's shadow (*pāta bimba*) are given for duration of a *nakṣatra* (*nakṣatrabhoga*) over the range from 56 to 66 *ghaṭi*s. Also the corresponding moon's motion is given in the last row. Note that the word *pāta* is used for the shadow and not for the moon's node.

In Table 8.5, the angular diameters of the sun (*ravi bimba*) and correction to shadow diameter (*pāta bimbas*) are given for 12 *saṅkrānti*s. Corresponding motion the sun (*ravi bhukti*) is also given in the last row.

PROCEDURE FOR FINDING DIAMETERS ACCORDING TO THE MAKARANDASARIṆĪ

The following procedure is given in the *Makarandasariṇī* for finding diameters:

1. Find the *gataeṣya ghaṭī* (duration) of the running *nakṣatra* at *parvānta* (full moon day or new moon day).
2. Consider the entry in the column headed by the number represented by *gataeṣya ghaṭī* of *nakṣatra* (taking only integer part of *ghaṭī*) corresponding to the row of *candra bimba*.
3. Find the entry in the next column (next to the column headed

by *gataeṣya ghaṭī* of *nakṣatra*) corresponding to the row of *candra bimba*. This entry is *agrimāṅka*.

4. Find the difference called *agrimāntara* between the above two entries obtained from steps 2 and 3. Now the remaining fractional part of *gataeṣya ghaṭī* or *vighaṭī* is to be multiplied by this difference *agrimāntara* and divided by 60. Add or subtract the result obtained to the first value or from *agrimāṅka* (the entries obtained in steps 2 and 3) respectively.
5. The above result gives angular diameter of the moon (*candra bimba*) in *aṅgula*s.
6. Similarly, we find the diameter of earth's shadow-cone (*bhūbhābimbam*) using Table 8.4 following the same procedure. In this case, instead of *candra bimba* row, the values corresponding to the row of *pāta bimba*s are to be considered. This diameter of shadow-cone should be corrected using Table 8.5 as explained in the following steps.
7. Find the *saṅkrānti* at *parvānta* by calculating the sun's position expressed in *rāśi, aṁśa, kalā* and *vikalā*s. Obtain *pāta bimba* value corresponding to running *saṅkrānti* and to the next *saṅkrānti* using Table 8.5 and take the difference between the two values.
8. Multiply the above difference between two values by *aṁśa*, *kalā* and *vikalā*s (degree, minutes and seconds) of the sun's position considered in step 7 and divide the product by 30. This result will be in *aṅgula*s.
9. Add the above result to the *pāta bimba* value corresponding to running *saṅkrānti* obtained in step 7. This is the correction factor to be added to the *pāta bimba* obtained using Table 8.5 in step 6 to get corrected angular diameter of earth's shadow (*bhūbhābimbam*).
10. Same procedure is followed to find the diameter of the sun using Table 8.5 corresponding the *saṅkrānti* at new moon on the day of solar eclipse.

Example: Given date: Śaka 1534 Vaiśākha *śuddha* 15, Monday.

For the above given date:

Parvānta ghaṭī = 54 | 40 *ghataeṣya ghaṭī* (duration) of *Anurādhā nakṣatra*

= 58 | 36 *ghaṭi*

True position of the sun = $1^R 6°30'37''$

True position of the moon = $7^R 6°34'35''$

Rāhu = $1^R 14°18'11''$.

To find diameter of the moon: Now from Table 8.4 the entry corresponding to *candra bimba* row in the column headed by *gataeṣya ghaṭī* number 58 is 11 | 10.

The entry corresponding to *candra bimba* row in the column headed by next number 59 (*agrimāṅka*) is 10 | 59.

The difference *agrimāntara* = 10 | 59 – 11 | 10 = – 0 | 11

The remaining fractional part of *gataeṣya ghaṭī* or *vighaṭī* = 36

Now, (difference × remainder)/60 = [(–11) × 36]/60 = – 396/60

= – 6.6 = – 6 | 36

= – 0 | 6 | 36 *aṅgula*s

Diameter of the moon (*candra bimba*) = 11 | 10 + (– 0 | 6 | 36)

= 11 | 4 *aṅgula*s.

To find diameter of the earth's shadow-cone: From Table 8.4 the entry corresponding to *pāta bimba* row in the column headed by *gataeṣya ghaṭī* number 58 is 28 | 16.

The entry corresponding to *pāta bimba* row in the column headed by next number 59 (*agrimāṅka*) is 27 | 38.

The difference = 27 | 38 – 28 | 16 = – 0 | 38

(difference × remainder)/60 = [(– 38) × 36]/60 = – 22 | 48 ≈ 22

Diameter of the shadow-cone (*bhūbhābimbam*) = 28 | 16 + (– 0 | 38)

= 27 | 54

Correction to *bhūbhābimbam* using Table 8.5:

True position of the sun at *parvānta* = $1^R 6°30'37''$

The number 1 in the *rāśi* position indicates that the current *saṅkrānti* is Vṛṣabha and the remaining degree, etc. in the sun's position is 6°30'37''.

Now, the entries in the column of Vṛṣabha and Mithuna Saṅkrānti corresponding to *pāta bimba* are 0 | 31 and 0 | 37 respectively. The difference = 0 | 37 – 0 | 31 = 0 | 6.

(difference × remainder)/30 = (6 × 6°30'37")/30

= 1.3020556 ≈ 1 = 0 | 1 *aṅgula*s

(ignoring the fraction)

Correction to *bhūbhābimbam* = 0 | 31 + 0 | 1 = 0 | 32 *aṅgula*s

Corrected *bhūbhābimbam* = 27 | 54 + 0 | 32 = 28 | 26 *aṅgula*s

Example: To find diameter of the sun: Given date: Śaka 1532 Mārgaśira *kṛṣṇa*, 30 Wednesday.

Parvānta ghaṭī = 11 | 59 True Sun = $8^R5°26'20"$.

The number 8 in the *rāśi* position indicates that the current *saṅkrānti* is Dhanu and the next is Makara. The entries in the column headed by Dhanu and Makara corresponding *ravi bimba* in Table 8.5 are 11 | 14 and 11 | 15 respectively.

Their difference = 11 | 15 – 11 | 14 = 0 | 1

Now the sun's diameter = 11 | 14 + (difference × remainder)/30

= 11 | 14 + (0 | 1 × 5°26'20")/30

= 11 | 14 + 0 | 10 | 52

= 11 | 24 | 52 ≈ 11 | 25 *aṅgula*s.

Remark: The dates given in the above examples are taken from Viśvanātha's commentary on the *Makarandasariṇī*.

Conclusion

In the present paper, we have discussed the procedures for determining *ahargaṇa*, true positions of the star planets and the angular diameters of the sun, the moon and the earth's shadow according to the *Makarandasāriṇī* in detail with concrete examples. It is shown how easy it is to convert a given traditional lunar calendar date to Kali days using *vallī* components of the *Makarandasāriṇī*.

Interestingly, the *Makarandasāriṇī* simplifies the procedure for computation of true planets, composing separate tables for *mandaphala* by consolidating the two conventional ways of applying *manda* equation twice. Also it gives the procedure for obtaining the angular diameters using the total duration of the running *nakṣatra* and the sun's *saṅkrānti* taking the readily available values from the traditional almanac (*pañcāṅga*).

Acknowledgement

We express our indebtedness to the History of Science Division, Indian National Science Academy (INSA), New Delhi, for sponsoring the research project for three of us (S.K. Uma, Padmaja Venugopal and Rupa K.) under which the present paper is prepared.

References

Bag, A.K., 1979, *Mathematics in Ancient and Medieval India*, pp. 257-59, Varanasi: Chowkamba Orientalia.

——— 2002, "Ahargaṇa and Weekdays as per Modern Sūryasiddhānta", *Indian Journal of History of Science*, 36(1-2): 55-63.

Grahalāghavam of Gaṇeśa Daivajña, S. Balachandra Rao and S.K. Uma, Eng. Exposition, Math. Notes, etc., *Indian Journal of History of Science*, 41 (1-4): 2006, S89 and S91.

Jyotīrmimāṁsā of Nīlakaṇṭha Somayāji, ed. K.V. Sarma, Hoshiarpur: Vishveshwaranand Vishwabandhu Institute of Sanskrit & Indological Studies, 1977.

Karaṇakutūhalam of Bhāskara II, S. Balachandra Rao and S.K. Uma, Eng. Tr. with Notes and Appendices, *Indian Journal of History of Science*, 42.1-2 (2007), 43.1 & 3 (2008), p. S18.

Makarandasāriṇī, comm. by Acharya Ramajanma Mishra, Varanasi: Madālasā Publications, 1982.

Makarandasāriṇī, comm. by Sri Gangadhara Tandan, Bombay: Sri Venkateshwara Press 1945.

Makarandaprakāśa, Pt. Lasanlala Jha, Varanasi: Chaukhamba Surabharati Prakashan, 1998.

Pingree, David, 1968, "Sanskrit Astronomical Tables in the United States (SATIUS)", *Trans. of the Am. Phil. Soc.*, 58(3): 1-77.

——— 1973, *Sanskrit Astronomical Tables in England (SATE)*, Madras: The Kuppuswami Sastri Research Institute.

Rai, R.N., 1972, "Calculation of Ahargaṇa in the Vaṭeśvara Siddhānta", *IJHS*, 7(1): 27-37.

Rao, S. Balachandra, 2000, *Ancient Indian Astronomy: Planetary Positions and Eclipses,* Delhi: B.R. Publishing Corp.

———, 2005, *Indian Mathematics and Astronomy: Some Landmarks* (rev. 3rd edn, 6th print), Bangalore: Bhavan's Gandhi Centre of Science & Human Values.

———, 2016, *Indian Astronomy: Concepts and Procedures*, Bengaluru: M.P. Birla Institute of Management.

Rao, S. Balachandra and Padmaja Venugopal, 2009, *Transits and Occultations in Indian Astronomy*, Bangalore: Bhavan's Gandhi Centre of Science & Human Values.

Rao. S. Balachandra, K. Uma and Padmaja Venugopal, 2004 "Mean Planetary Positions According to Grahalāghavam", *Indian Journal of History of Science*, 39(4): 441-66.

Rupa, K., Padmaja Venugopal and S. Balachandra Rao, 2013, "An Analysis of the Mandaphala Tables of Makaranda and Revision of Parameters", *Ganita Bharatī*, 35(1-2): 221-40.

Rupa, K. Padmaja Venugopal and S. Balachandra Rao, 2014 "Makarandasāriṇī and Allied Saurapakṣa Tables: A Study", *Indian Journal of History of Science*, 49(2): 186-208.

Sodāharaṇa Makarandasāriṇī with Viśvanātha Daivajña's Commentary *Udāharaṇam,* Bombay: Śrī Venkateśvara Press, 1913.

Uma K. and S. Balachandra Rao, 2018, "Ahargaṇa in Makarandasāriṇī and Other Indian Astronomical Texts", *Indian Journal of History of Science*, 53(1): 16-32.

9

Manuscripts on Indian Mathematics

K. Bhuvaneswari

As the famous quote states:

> यथा शिखा मयूराणां नागाणां मणयो यथा ।
> तद्वद्वेदाङ्गसास्त्राणां गणितं मूर्ध्नि स्थितम् ।।
>
> – *Vedāṅga Jyotiṣa* v. 4

> Like the crests on the head of peacocks, like the gems on the heads of the cobra, mathematics is at the top of the Vedāṅgaśāstras.

Also, Mahāvīrācārya, in his *Gaṇitasāra-Saṁgraha* (I.16), states:

> बहुभिर्विप्रलापैः किं त्रैलोक्ये सचराचरे ।
> यत्किञ्चिद्वस्तु तत्सर्वं गणितेन विनानहि।।

> Whatever is there is all the three worlds, which are possessed of moving and non-moving beings, all that indeed cannot exist as apart from mathematics.

The significance of mathematics and its application in all walks of life was well realized by the great seers of the Vedic period, the poets of the classical period and kings of the past. In olden days, mathematics was included in the Jyotiṣaśāstra, which is one of the six Vedāṅgas. Hence, numerous works were written on mathematics, astrology and astronomy, the three major divisions of the Jyotiṣaśāstra.

Also, India has had a continuous lineage of mathematicians and they had been pioneers in introducing various mathematical concepts. Their seminal contributions had been the foundations of many branches of mathematics and have led to further developments. Their works are generally in *padya* (verse) form and are mostly accompanied by prose commentaries of the author or other scholars. There are many commentaries for each work by different scholars.

As Sanskrit was the language of the learned and the medium of higher education since the Vedic period, these texts and commentaries are in Sanskrit, mostly available as manuscripts either in palm leaf or in paper.

Manuscripts are carriers of culture and also act as link between the present and the past. Hence, it becomes necessary that these manuscripts are preserved well for the benefit of the posterity. Since they are subject to easy destruction, preserving the old text is always a difficult task. There had been *lipikāra*s (scribes) appointed by the kings to make copies of the available manuscripts, in the script of the existing era. In spite of all such efforts, most of the manuscripts available now are not older than 600 years. A few are 1,000 years old and some fragments of manuscripts are even, it is said, 2,000 years old.

These manuscripts are preserved in various manuscript libraries in and outside India. There are more than 300 manuscript libraries both small and large all over India; and India is estimated to have nearly three crore manuscripts.

Some of the major manuscripts libraries in India are:

1. The Sarasvati Bhavan Library of the Government Sanskrit College, Benares now attached to Sampurnananda Sanskrit University, established in 1791.
2. Tanjore Maharaja Serfoji's Sarasvati Mahal Library (TMSSML), established during the early part of nineteenth century.
3. Ranvir Sanskrit Residential Institute, Jammu, established in 1857.

4. Government Oriental Manuscripts Library (GOML), Chennai, taking care of manuscripts since 1870.
5. Adyar Library and Research Centre, Chennai, established in 1886.
6. Government Oriental Library, Mysore, established in 1891.
7. Central Library, Baroda, preserving Sanskrit manuscripts since 1893.
8. Bhandarkar Oriental Research Institute, Bombay, established in 1917.
9. Maharaja Palace Library, Trivandrum, established between 1817 and 1827 and later amalgamated into Oriental Manuscript Library, Kerala University in 1937.
10. Sri Venkateswara University Oriental Research Institute, established by Sri Venkateswara University in 1939.
11. Kuppuswami Sastri Research Institute, Chennai, established in 1944.

As far as Tamil Nadu is concerned, the GOML, Adyar Library & Research Centre and the TMSSML have a big collections of manuscripts. In view of the vastness of the manuscripts on Jyotiṣa, I restrict myself to the manuscripts on mathematics alone, available in Tamil Nadu, for critical edition, publication and further research.

Līlāvatī of Bhāskara II of Twelfth Century

It is the most famous treatise in ancient Indian mathematics. It is a part of the much larger treatise *Siddhānta-Śiromaṇi*. It is a work on arithmetic.

There are as many as sixty-eight commentaries on this work. But only a few (of Gaṇeśa, Mahīdhara and Śaṅkaranārāyaṇa) have been edited and printed so far. Some of the commentaries available for further research are:

GOML – MD-13484; Paper MS; Devanāgarī; Complete

कर्मप्रदीपिका/ *Karmapradīpikā* by Nārāyaṇa of sixteenth century.

It begins with *guruvandanam* to Bhāskara and Āryabhaṭa

प्रणम्य भास्करं देवमाचार्यार्यभटं तथा ।
व्याख्या विलिख्यते लीलावत्याः कर्मप्रदीपिका ।।

The work ends again with salutations to Āryabhaṭa and also carries the name of the author as well.

एतन्नारायणाख्येन रचितं कर्मदीपकम् ।
सन्तिष्ठतु परं लोके नमाम्यार्यभटं सदा ।।

GOML – MD-13486; Palm Leaf; Grantha; Incomplete

लीलावतीविलासः/Lilāvatīvilāsa by Raṅganātha of fifteenth century.

Stating the purpose of the work he is about to compose, the author also mentions his name in the following *śloka* in the beginning of his work:

लीलावतीवुलासोऽयं बालानन्दैक कारणम् ।
लिख्यते रङ्गनाथेन सोपपत्या निरर्थकम् ।।

This manuscript is slightly injured and breaks off in the Khaṭa Vyavahāra.

GOML – MT-3938; Palm Leaf; Grantha; Slightly Injured; Incomplete

लीलावतिविलासः/Lilāvatīvilāsa by an anonymous author. It begins with salutations to his *guru* and Goddess Sarasvatī.

नत्वा गुरुचरणाम्बुजमम्बां वागीश्वरीं च वाक्सिद्ध्यै ।
लीलावतीविलासं रचये रसिकजनमौलिसन्तुष्ट्यै ।।

The author has not mentioned the name of his *guru*. Since the text is incomplete there is no colophon to find the name of the author or the name of his *guru*.

GOML – MT-5160; Paper MS; Devanāgarī; Complete

लीलावतीव्याख्या/ Lilāvatīvyākhyā by Parameśvara of fourteenth century, pupil of Rudra.

After offering salutations to gods, praying for the removal of obstacles and successful completion of the work in the first two *śloka*s, the author mentions his name as Parameśvara:

लक्ष्मीभूविलसत्पार्श्वः सहस्रादित्यसंनिभः ।
ज्ञानमूर्तिरनाद्यन्तो हरिरिष्टं ददातु नः ।।

प्रणमामि गणेशानं पार्वत्या अङ्कसंस्थितम् ।
वागीश्वरीमपि तथा श्रीरुद्रं च कृपानिधिम् ।।

नीलायाः सागरस्यापि तीरस्थः *परमेश्वरः* ।
व्याखयानमस्मै बालाय लीलावत्याः करोम्यहम् ।।

A copy of this MS is available in Adyar Library and Research Institute.

GOML – MT-5244; Palm Leaf MS; Devanāgarī; Complete

सर्वबोधिनी व्याख्या/*Sarvabodhinīvyākhyā* by Mahāpātra Śrīdhara of seventeenth-eighteenth century, s/o Nima and Gaurī.

It begins with an invocation to God Gaṇeśa and Goddess Sarasvatī:

भक्तानुग्रहकाम्यया निजतनौ रागातिरेकं दध–
द्विघ्नध्वान्त नितान्तशान्तिकरणे सूर्यायमाणस्थितिः ।
विभ्राणो रदनाक्ष सूत्रपरशूम् हस्ताम्बुजैर्मोदकं
कामं पूरयतु प्रकाममखिलं विघ्नेश्वरो मे सदा ।।
शितितर रुचिं सर्वजडत्वध्वंसकारिणीम् ।
सर्वामरार्चितपदामहं वन्दे सरस्वतीम् ।।

The author mentions his name and the name of the commentary in the following *śloka:*

गुरुपादप्रसादेन श्रीधरेण द्विजन्मना ।
पाटीगणित टीकेयं क्रियते *सर्वबोधिनी* ।।

In the ending verses, the author mentions about his father and mother:

विद्याषट् त्रयवेदिनो मखशतैः पूतस्य तस्यान्वये
कश्चिद्दैवविदां वरोऽजनि *महापात्रो निमाख्यः* कविः ।
गौर्यां मातरि हन्त तेन जनितः प्रीति प्रवृद्ध्यै सतां
पाटीं गाणितकीमटीकातरां विद्वद्वरः श्रीधरः ।।

And the work ends with *maṅgala śloka*:

टीका विज्ञजनानन्ददायिनी *सर्वबोधिनी* ।
जयतु व्यक्तगणितान्वयन्यायक्रमोज्ज्वला ।।

TMSSML – 11592; A Paper MS; Devanāgarī; Incomplete

लीलावती व्याख्या/Lilāvatīvyākhyā of Keśava of fourteenth century.

The name of the author is known from the title page. The colophon is simple to contain only the name of the chapter.

Bījagaṇita of Bhāskara II of Twelfth Century

It is another famous work of Bāskara II, wherein algebra is dealt with. This is the second part of the treatise *Siddhānta-Śiromaṇi*.

COMMENTARY ON BĪJAGAṆITA

There are six commentaries on the *Bījagaṇita*. Of these, the *Bījapallava* of Kṛṣṇa Daivajña has been already studied by Sita Sundar Ram and published by The Kuppuswami Sastri Research Institute in 2012.

Of the others, the available commentary is the *Bījagaṇita-vyākhyā* of Sūryadāsa.

GOML – MD-13462; Palm Leaf; Grantha; Slightly Injured; Incomplete

बीजगणितव्याख्या/Bījagaṇita-vyākhyā by Sūryadāsa, s/o Jñānarāja.

The author gives details about him in the beginning as, he is a pupil of Jñānarāja and also his son:

छन्दोलङ्कृतिकाव्यनाटकमहासङ्गीतशास्त्रार्थवित् ।
तं वन्दे निजतातमुत्तमगुणं श्रीज्ञानराजं गुरुम् ।।

and mentions his name in the following verse:

विमुग्धानां प्रीत्यादृतसदयचेताः परिमिता
मिमां व्याख्यानार्थं सपदि रचये सूर्यगणकः ।।

It is stated that the entire composition is set in Upendravajrā metre.

... उपेन्द्रवज्रावृत्तेन ग्रन्थतो निबध्नाति ।

It is clear that they had not only been masters in their fields but also proficient in Sanskrit Grammar and Śāstras.

A part of this commentary is published by Pushpakumari Jain and a part is under preparation by Sita Sundar Ram for Indian National Science Academy, New Delhi.

Mahābhāskarīya of Bhāskara I of Seventh Century

It is an astromomical treatise of Bhāskara I. Though an astronomical treatise, it contains certain mathematical derivations and approximate values of trigonometric sines.

COMMENTARIES ON MAHĀBHĀSKARĪYA

There are two commentaries on the *Mahābhāskarīya*. The *Mahābhāskarīya-vyākhyā-karmadīpikā* by Parameśvara is already edited and printed. The other commentary available is *Prayogaracanā*.

GOML – MT-3034; Paper MS; Devanāgarī; Complete

महाभास्करीयव्याख्या–प्रयोगरचना/ Mahābhāskarīya-vyākhya-Prayogaracanā – an anonymous work.

It begins with an invocation to Lord Śiva:

प्रणमत(पर)शिवमनिशं तं यं सद् ब्रह्मवादिनः प्राहुः ।
यस्य च विभूतिरेषा क्षित्यादीनां प्रकाशाख्या ।।

क्रियते प्रयोगरचना गुरुप्रसादेन भासकरीयस्य ।
यैषा प्रदीपिकेव प्रकाशयित्री च सूक्ष्मवस्तूनि ।।

The name of the author is neither stated at the end of the work nor in colophon.

Kuṭṭākāraśiromaṇi

Kuṭṭākāra is a kind of mathematical calculation and this work deals with it.

There are two works in this name by different authors. One is the *Kuṭṭākāraśiromani-vyākhyā* of Devarāja with self-commentary which is already printed. The other is the *Kuṭṭākāraśiromaṇi* by Veṅkaṭādri.

TMSSML – 11354; Palm Leaf MS; Grantha; Incomplete.

कुट्टाकारशिरोमणि: सटीका /Kuṭṭākāraśiromaṇi-saṭīkā.

This is a commentary on *Kuṭṭākāraśiromaṇih* of Veṅkāṭadri of seventeenth century, by an unknown author.

This author of *Kuṭṭākāraśiromaṇi* seems to be the same as the *Bhūgola Veṅkaṭeśa*, whose first verse is similar in all his works.

It begins as:

स्थूलसूक्ष्मपरिकल्पिताखिलं वर्गभेदपरिकर्मसंकुलम् ।
सौरमानमुख कालकारणं शेषपर्वतशिखामणिं भजे ।।

This work of Veṅkaṭādri is dedicated to his patron king, the fourth Nāyaka ruler of Tanjore.

It is stated in the following *śloka* at the end of the work:

स्वस्तिरस्तु ते *विजयराघवभूमिपाल* ....।।

एकस्मिन् वासरे यस्तु महादानमशेषतः ।
अकरोत् षोडशमितं भूसुरेभ्यो महामतिः ।।

तन्नामगुणेनैव कुट्टाकारशिरोमणौ ।
उपोद्घातः परिच्छेदः कृतोऽयं *वेङ्कटाद्रिणा* ।।

The Āryabhaṭīya of Āryabhaṭa I of Sixth Century

The *Āryabhaṭīya* is a work on astronomy and mathematics by Āryabhaṭa I. The mathematics portion contains thirty-three *sūtra*s.

COMMENTAY ON ĀRYABHAṬĪYA

One commentary on this work the *Bhaṭaprakāśa* is available.

GOML – MT-3862; Palm Leaf MS; Grantha; Slightly Injured; Incomplete

भटप्रकाश–आर्यभटसूत्रार्थप्रकाशिका/ Bhaṭaprakāśa-Āryabhaṭasūtrārthaprakāśikā

This commentary on the *Āryabhaṭīya* is by Sūryadevayajvan, s/o Bālāditya. After offering salutations to Lord Viṣṇu in the first stanza, the author mentions his name and about the work.

त्रिस्कन्धार्थविदा सम्यक् *सूर्यदेव यज्वना* ।
आचार्यार्यभट प्रोक्तसूत्रार्थोऽत्र प्रकाश्यते ।।

Lexicons

There are some interesting lexicons available in manuscript form. Each text is special in its own way.

GOML – MD-13601; Palm Leaf; Grantha; Complete

अङ्कनिघण्टुः/ Aṅkanighaṇṭu – an anonymous work.

A lexicon of synonymous terms for denoting the numbers one to nine and zero. In this work, numbers are represented by words denoting objects in the natural world and religious world. This form of representing numbers is called *bhūtasaṁkhya*.

It begins as:

शशी सोमश्शशाङ्कश्च इन्दुश्चन्द्रश्च रूपकम् । ....

And ends as:

आकाशं गगनं शून्यं व्योम पुष्करमम्बरम् ।
खंच वायुपदं तच्च श्रीकरानन्तभीकरम् ।।

GOML – MD-13603; Palm Leaf; Grantha; Complete

अङ्कनिघण्टुः /Aṅkanighaṇṭu – an anonymous work.

The words denoting the numbers above nine are given.

The beginning reads like this:

एकस्य भूरूपशशाङ्कनामान्यूहन्ति पूर्वं गणनाविधिज्ञाः ।

And ends as:

द्वाविंशतेराक्रुतिस्तु चतुर्विंशजनाभिधः ।
तत्पावनी पञ्चविंशत्या षड्विंशत्सप्तविंशतिः ।

GOML – MD-14018; Palm Leaf; Grantha; Complete

अङ्कनघिण्टुः/Aṅkanighaṇṭu – an anonymous work.

This is similar to the above works. It deals with the pace value.

Its beginning is:

प्रथममेकस्थानं द्वितीयं दशसंज्ञिकम् ।
तृतीयं शतमित्येतच्चतुर्थं तु सहस्रकम् ॥

And ends as:

महामृतं त्रयस्त्रिंशद्भूरिवेदाग्निसंज्ञिकम् ।
पञ्चत्रिंशमहाभूरि षड्त्रिंशायुतभूरि च ॥

GOML – MD-13407; Paper MS; Telegu; Complete

गणितप्रकाशिका/Gaṇitaprakāśikā – an anonymous work.

It contains alphabetical list of mathematical terms with Telugu meaning.

It begins as:

अङ्कम्, अङ्कपाशम्, अङ्कुलम्,अन्तम्, ...।

And ends as:

क्षुण्णम्, क्षेत्रम्, क्षेपम्, क्षिप्तम् ॥

Pañcāṅgas

There are many works on the preparation of *pañcāṅga*s. Of them the following deals with the mathematical calculations required for the preparation of almanacs.

TMSSML – 11655; Palm Leaf; Telegu; Slightly Injured; Incomplete

पञ्चाङ्गगणितम्/ Pañcāṅgagaṇitam – an anonymous work.

It deals with certain mathematical calculations required for the preparation of almanacs.

GOML – MD-13447; Paper MS; Devanāgarī; Complete

पञ्चाङ्गगणितविषयः/ Pañcāṅgagaṇitaviṣayaḥ

It also deals with certain calculations required in the preparation of the Hindu calendars.

GOML – MT-1042; Palm Leaf MS; Telegu; Complete

प्रतिभागः /Pratibhāgaḥ – an anonymous treatise.

It is a short manual containing the rules for computing various particulars required for the preparation of Hindu almanacs.

There are many other works, the names of which are specified in the book *A Bibliography on Sanskrit Works on Astronomy and Mathematics* by S.N. Sen, the details of which are not available in GOML. A few of them are highlighted below.

GOML – MD-16787; Palm Leaf MS; Grantha; Incomplete

गणितग्रन्थः/Gaṇitagranthaḥ –

It is an anonymous work on arithmetic dealing with commercial accounts with examples.

A copy of this MS is also kept in Adyar Library and Research Institute.

GOML – MT-3943; Palm Leaf MS; Grantha; Complete

गणितसंग्रहः/ Gaṇitasaṁgrahaḥ

It is a commentary on the *Sūryasiddhānta* by an unknown author. The work is named as the *Siddhānta-Saṁgraha* in its colophon.

A copy of this MS is also with in Adyar Library and Research Institute.

Catalogue Raisonné of Oriental Manuscripts by Taylor-1548 (now in GOML)

गणितशास्त्रः / Gaṇitaśāstraḥ of Mahārāja – It is a work on mathematics.

Catalogue Raisonné of Oriental Manuscripts by Taylor-1548 (now in GOML)

क्षेत्रगणितसारः/Kṣetragaṇitasāraḥ – An anonymous work on geometry.

GOML – MT-3864

लीलावती व्याख्या /Līlāvatī-vyākhyā of Kāma – It is a commentary on the *Līlāvatī*.

There are some more works on mathematics available in manuscript form in the Adyar Library and Research Institute, the details of which are furnished below:

- *75262-b – आर्यभटीयव्याख्या–गीतिप्रकाशः/ Āryabhaṭīyavyākhya – Gītiprakāśaḥ – Palm Leaf MS; Malayalam; Incomplete.*

It is a commentary on the *Āryabhaṭīya* by an unknown author.

- *PM1299-b – आर्याभटीयविषयानुक्रमणिका / Āryabhaṭīyaviṣayānukramaṇikā – Paper MS; Telegu; Complete.*

It is also a commentary on the *Āryabhaṭīya* by an unknown author.

- *PM1300 – कौतुकलीलावती /Kautukalīlāvatī* by Rāmacandra – *Paper MS; Devanāgarī; Complete.*

- *67736 – गणितत्रिबोधः/ Gaṇitatribodhaḥ – Palm Leaf MS; Grantha; Complete; Damaged.*

It is a work by an anonymous author.

- *75263-b – गणितविषयः/ Gaṇitaviṣayaḥ – Palm Leaf MS; Grantha; Incomplete.*

It is a *karaṇa* text by an anonymous author.

- 68537-a – *गणितसङ्ख्या / Gaṇitasaṅkhyāḥ – Palm Leaf MS; Malayalam; Incomplete; Damaged.*

It is an anonymous work.

There are many other works on Mathematics, available as manuscripts all over India in various manuscript libraries. It is indeed a matter of pride to learn that our country has a strong mathematical heritage. In fact, every Indian should know about the rich legacy of our ancient mathematicians. Hence, it is important that these works which are in manuscripts form are to be taken care of, preserved to prevent from deterioration, catalogued properly, edited and studied diligently.

10

Study of the Ancient Manuscript Mahādevī Sārīṇī

B.S. Shubha
B.S. Shylaja
P. Vinay

Abstract: The natural units of time such as day, month and year, that are essential for human activities are mostly guided by the movements of heavenly bodies. The astronomical tables known as *sāriṇī*, *koṣṭaka* and *karaṇa* are usually short collections of necessary data and rules for standard astronomical calculations. Theoretical treatises deal with a comprehensive exposition of astronomy frequently containing descriptions of its underlying geometric models. The *Mahādevī Sāriṇī* by author Mahādeva is one among these. The study shows that the computed positions are in fair agreement including the retrograde motion.

Introduction

VEDĀṄGA Jyotiṣa forms a branch of the Vedas which deals with the Indian astronomy. The science of astronomy developed in India with naked eye observations from time immemorial is fascinating. Many astronomers and their works have remained unknown to us. Some of the works have to be edited and presented to the scholars of next generations. The pioneering efforts are initiated by R. Shyamashastri from Mysore, Sudhakar Dvivedi from Varanasi, T.S. Kuppana Shastri from Chennai among others. Natural units of time, day and year are determined by the movements of heavenly bodies. The astronomical tables known as *sāriṇī*, *koṣṭaka* and *karaṇa* provide these quantities. The *Mahādevī Sāriṇī* was one such table very widely used earlier.

Mahādevī

From the opening verses of the commentary on the *Mahādevi Sāriṇī*, we come to know that it was started by astronomer Cakreśvara. And then the incomplete work was completed by Mahādeva. His father's name was Paraśurāma and Mādhava was his grandfather's name. There is a work named *Jātakasāra* written in both Sanskrit and Gujarati which has recommended the calculation of planet's positions from the *Mahādevī Sāriṇī* (Subbarayappa and Sarma 1985).

Mahādevī Sariṇī

The *Sāriṇī* was written during the epochal year 1238 S.S. corresponding to 1316 CE. Mahādeva has adopted four and a half as the *palabha* for calculating the ascensional difference.

The number of tables for planets itself appears to be very uniquely arranged. With reference to the sun, we are studying the position of Jupiter as provided by the *Sāriṇī* at fixed intervals as decided by the speed of movement of the planet.

The values of the movements of the planet are recorded in the manuscript. The true longitudes of the planets are available in it. Tables use layout to enhance the mathematical usage and highlight the phase. The initial position of the sun is set at Aries 0°. There are 60 tables for each planet. $360/\lambda = 60$, where $\lambda = 0$ to 6° (Neugebauer and Pingree 1967). In this study we interpolate the positions of Jupiter as provided by the *Sāriṇī*. We have chosen 1311 CE from the second table of the *Sāriṇī*, so as to match with the positions provided by the software (cosine kitty.com), which match fairly well. This applies for retrograde motion also.

The first row is numbered 1 to 27, which are the *avadhi*s, interval of 14 days $(26 \times 14) = 364$ days. The next row gives true longitude of the planet, followed by interpolation row. The next row gives

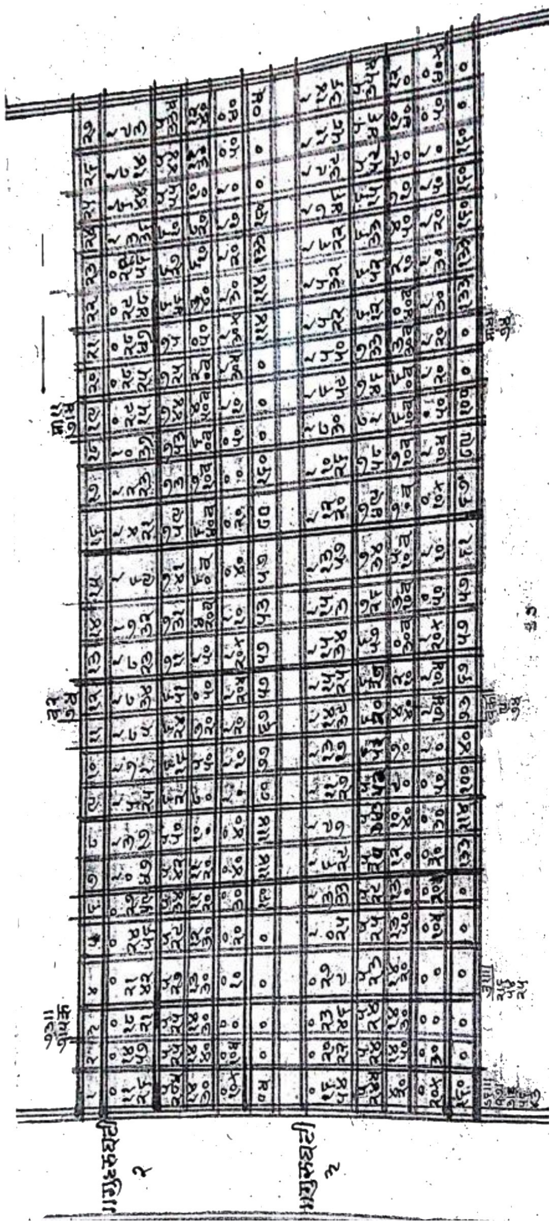


fig. 10.1: A typical table for Jupiter in the *Sāriṇī* the daily velocity followed by its interpolation. The next row has a value of 800 minutes. There is no clarity in the manuscript about its interpretation. The last row gives the planetary phases such as *vakra*, *mārga*, *aṣṭapaścima*, *aṣṭapūrva*, *udayapaścima* and *udayapūrva*. That tells about the synodic phases of the planet (Agathe Keller et al., online recources Id: halshs 01006137).

The longitudes are converted to right ascension with the help of spherical trigonometric equations (Hari 2006). The equation used is $\tan \alpha = \tan \lambda \cos \varepsilon$. Where λ = sidereal longitude, α = right ascension and $\varepsilon = 23.5$.

Discussion

Calculated values of right ascension from the *Sāriṇī* are compared with the values by the software and the results are shown in *figs* 10.2 to 10.7.

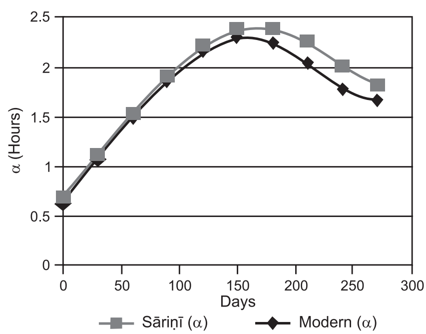


fig. 10.2: Right ascension for the year 1311 CE

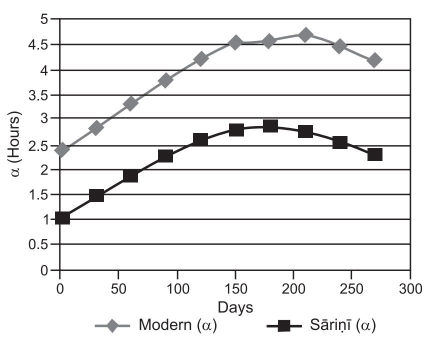


fig 10.3: Right ascension for the year 1312 CE.

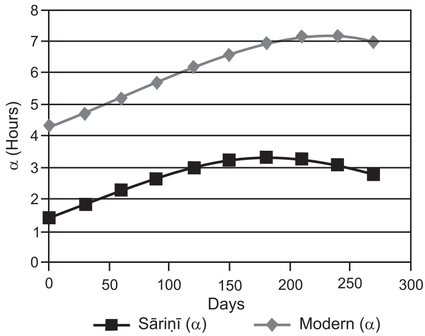


fig 10.4: Right ascension for the year 1313

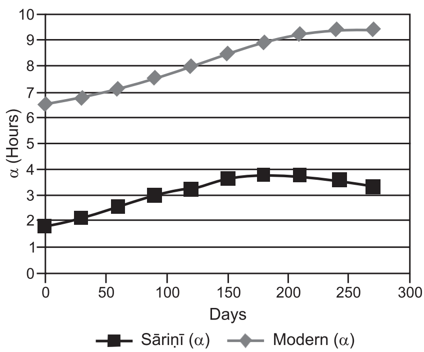


fig 10.5: Right ascension for the year 1314

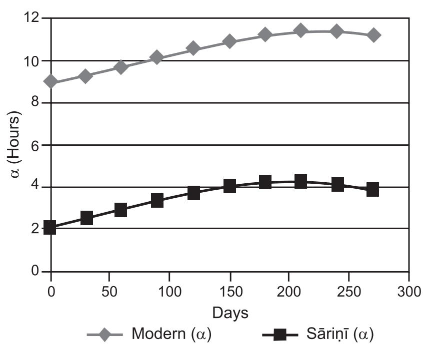


fig 10.6: Right ascension for the year 1315

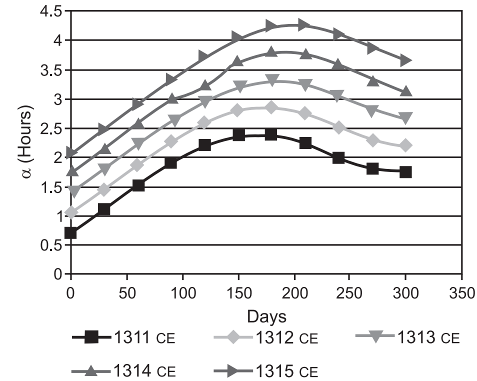


fig 10.7: Right ascension (year 1311 CE to 1315 CE)

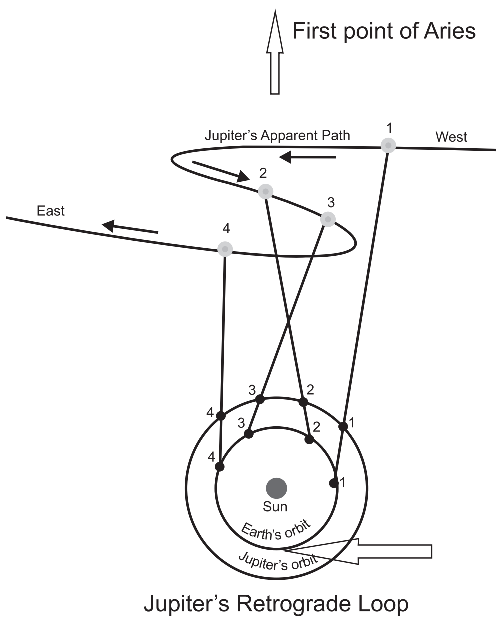


fig 10.8: Explanation of retrograde motion of the Jupiter. All the values of the *Sariṇī* are adjusted to the position of the earth indicated by the arrow corresponding to the sun and the Jupiter both in the first point of Aeries

1. The values computed by the software and the *Sarīṇī* vary within a degree. For every year position of Jupiter coincides with the first point of Aries.
2. The annual shift towards right is explained by the annual motion of Jupiter.
3. Only true longitudes are utilized for the study.
4. We have not done interpolation using other rows or column values.
5. The onset of retrograde motion exactly coincides with the note *vakra* in the last row of the manuscript. We are planning to get the precise time of onset of Jupiter using the interpolation.

Conclusion

The study shows that the computed positions of Jupiter are in fair agreement including the retrograde motion. While analysing this manuscript (Neugebauer and Pingree 1967). Pingree had attributed many scribal errors however we have not seen any in the case of Jupiter so far. We have just begun the study of *Sāriṇī*. The meaning of other rows has to be analysed and verified. The table also demonstrates another aspect, perhaps all these positions were verified by observations. However more number of *Sāriṇī* and their theory have to be studied and verified before commenting about this aspect.

Acknowledgement

We are thankful to staff of Bhandarkar Oriental Research Institute, Pune, for providing the necessary help in obtaining the manuscript. We are also thankful to The Samskrita Academy, Madras for providing us an opportunity to present my results on *Sāriṇī*.

References

Cosine.kitty.com (Online Resources)

Hari, Chandra, 2006, "Polar Longitudes of the Suryasiddhanta and

Hipparchus' Commentary", *Indian Journal of History of Science*, 41(1): 29-52.

Keller, Agathe, et al, "Numerical tables in Sanskrit sources" HAL Id: halshs - 01006137.

Mahesh Koolakodlu and Montelle Clemency, "Numerical Tables in Sanskrit Sources".

Neugebauer, O. and David Pingree, 1967, "The Astronomical Tables of Mahadeva", *Proceedings of the American Philosophical Society*.

Subbarayappa, B.V., and K.V. Sarma, 1985, *Indian Astronomy: A Source-book*, Bombay: Nehru Centre.

11

Fibonacci Sequence

History and Modern Applications

Vinod Mishra

Abstract: The variations of *mātrā-vṛtta*s form the sequence of numbers 1, 2, 3, 5, 8, 13, ..., now called Fibonacci sequence, is governed by the recurrence relation $F_n = F_{n-1} + F_{n-2}$; $n \geq 2$, u_0 u_1 = 1. It is part of combinatorial problems in Indian mathematics. The limit of the ratio between two successive Fibonacci numbers is often termed as the golden ratio, mean or proportion, viz.

$$\lim_{n \to \infty} \frac{f_n}{f_{n-1}} = \varphi = \frac{1+\sqrt{5}}{2}.$$

The paper inculcates historical development of Fibonacci sequence and its modern applications in science, engineering and medicine.

Keywords: Hemacandra–Virhaṅka sequence, Gopāla–Hemicandra sequence, metres, Fibonacci numbers or sequence; golden ratio, golden section, coding; DNA/RNA, Fibonacci polynomials.

Introduction to Fibonacci Numbers

Undoubtedly, well before the time of Italian Leonardo Fibonacci (1170–1250 CE) of Pisa, the concept of Fibonacci sequence was understood and applied in India in connection with metrical science by the legends Piṅgala (fl. 700 BCE – 100 CE), Bharata (fl. 100 BCE – 350 CE), Virahaṅka (fl. 600 – 800 CE), Gopāla (*c*.1135 CE) and Ācārya Hemacandra (1088 – 1173 CE). For detailed historical development of combinatorics, musical connection and Fibonacci like numbers, one may refer to Mishra (2002), Singh (1985), Shah (2010), Seshadri (2000), Kak (2004), Sen et al. (2008), etc.

This topic aims at fulfilling the gap between history of mathematics, and modern science and applications.

APPLICATIONS

Existing Fibonacci sequence and further extension of Fibonacci sequence to generalized Fibonacci sequence and Fibonacci polynomials lead to certain exciting applications in music, science, engineering and medicine:

Physical Science

- Mathematics (plane geometry: golden rectangle and isosceles triangle, regular pentagon and decagon; platonic solids: icosahedron, regular dodecahedron, octahedron, hexahedron and tetrahedron; Keplar triangle; solutions of integral and fractional order differential equations, integral equation, partial differential equation, difference equation, state space equation in dynamical system).
- Statistics (random process, Markov chain, set partition, correlation analysis).
- Physics (hydrogen bonds, chaos, superconductivity).
- Chemistry (quantum crystals, protein AB models, fatty acids).
- Astrophysics (pulsating stars, black holes).

Biology and Medicine

- Genetic coding, DNA/RNA structure, population dynamics, natural and artificial phyllotaxis, multicellular models, MRI

Music

- Musical harmony.

- Musical structures.

Engineering

- Crypto-communication (coding, mobile network security, elliptic curve cryptosystem).
- Signal processing include: Face detection evaluation, fashion and textile design, analog-to-digital converter design, traffic signal timing optimization, heart and perception-based biometrics, audio and speech sampling, barcode generation.
- Engineering (tribology, resisters, quantum computing, quantum phase transitions, photonics).

EUCLID'S THEOREM (ELEMENTS, *c*.300 BCE)
(Agaian and Gill III 2017; Agaian 2009).

Divide a line AB into two segments, a larger one CB and a smaller one AC such that $CB^2 = AB \times AC$, where $CB > AC$ and $AB = AC + CB$. Then $AB/AC = AB/CB$.

Letting $x = CB/AC$, $x^2 - x - 1 = 0$. The positive root implies $\varphi = 1.618$ and is called the golden ratio or proportion. Kepler later discovered that the golden ratio can be expressed as ratio of two consecutive Fibonacci numbers.

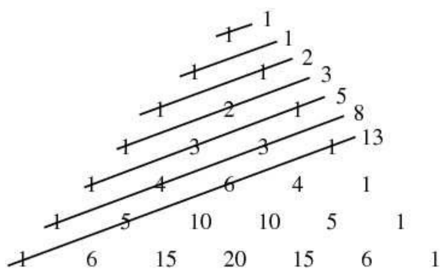
Hemacandra–Virahaṅka Sequence

The Jaina writer Ācārya Hemacandra (1088–1173 CE) studied the rhythms of Sanskrit poetry. Syllables in Sanskrit are either long or short. Long syllables have twice the length of short syllables. The question he asked is how many rhythm patterns with a given total length can be formed from short and long syllables?

Ācārya Hemacandra in his *Chandonuśāsana* (*c*.1150 CE), mentions the idea of the number of variations (patterns) of *mātrā-vṛtta*s. His rule is translated thus:

> Sum of the last and the last one but one numbers (of variations) is (the number of variations) of the *mātrā-vṛtta*s coming afterwards.
>
> – Meinke 2011

Number	F_{n-1}	F_{n-2}	F_n
0		0	0
1		1	1
2	1	0	1
3	1	1	2
4	2	1	3
5	3	2	5
6	5	3	8
..	..	..	..



Mount Meru

He continues:

> From amongst the numbers 1, 2, etc. those which are last and the last but one are added (and) the sum, kept thereafter, gives the number of variations of the *mātrā-vṛtta*s. For example, the sum of 2 and 1, the last and the last but one, is 3 (which) is kept afterwards and is the number of variations (of metre) having 3 *mātra*s. The sum of 3 and 2 is 5 (which) is kept afterwards and is the number of variations (of the metre) having 4 *mātrā*s. ... Thus: 1, 2, 3, 5, 8, 13, 21, 34 and so on.

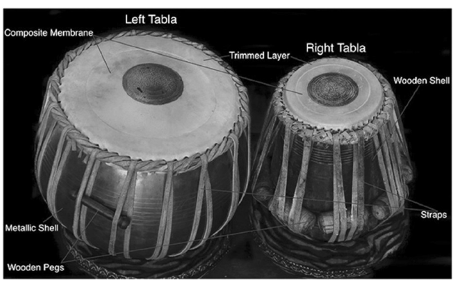
Mount Meru is called Yang Hui's triangle in Chinese terminology after Yang Hui (fl. 1238-98), Tartagalia's triangle after Italian Tartagalia (1499–1557) and Pascal's triangle in Western Europe due to Blaise Pascal (*Traite du triangle arithmerique*, 1655).

The sequence of numbers of patterns now called the Fibonacci sequence, after the Italian mathematician Fibonacci, whose work (*Liber Abaci*, *c.*1202; Book of Calculation) was published seventy years after Hemacandra. The numbers in the sequence are called Fibonacci numbers. Fibonacci introduced and popularized Hindu–

Arabic numeral system to Western countries (Europe) through *Liber Abaci*.

MUSIC CONNECTION

> The poetic metres Hemacandra studied have an analogue in music. Rhythm patterns are sequences of drum hits that overlay a steady pulse, or beat. Notes – groups of beats – play the role of syllables in poetry. Drummers hit on the first beat of a note and are silent on the following beats; the length of a note is the number of beats from one hit to the next.
>
> Returning to our musical question, the answer is that the number of rhythm patterns with length n is the sum of the number of patterns of length $n - 1$ and the patterns of length $n - 2$.
>
> – Hall 2008



Tablā (combination of pair of drums – *byan* (big) and *dayan* (small)) (Tiwari and Gupta 2017)

Bola – *dha* (1 beat) – time duration 1 or length 1

Bola – *thira kita* or *te te* (2 beats) – time duration 2 or length 2

Example: Different combination of 1 and 2 – to have metre (*chandaḥ*, composition) of length 4 beats (syllables):

Variations or Patterns of Length of Five Beats	
LL	2 + 2 = 4
SSL	1 + 1 + 2 = 4
LSS	2 + 1 + 1 = 4
SLS	1 + 2 + 1 = 4
SSSS	1 + 1 + 1 + 1 = 4
	Total 5

S – Short, L – Long

Example: Different combinations of 1 and 2 – to have metre (*chandaḥ*, composition) of length 5 beats (syllables):

Variations or Patterns of Length of Five Beats		
SLL	1 + 2 + 2 = 5	*dha thir kita*
LSL	2 + 1 + 2 = 5	*thit kita dha thir kita*
SSSL	1 + 1 + 1 + 2 = 5	
LLS	2 + 2 + 1 = 5	
SSLS	1 + 1 + 2 + 1 = 5	
SLSS	1 + 2 + 1 + 1 = 5	
LSSSS	2 + 1 + 1 + 1 + 1 = 5	
SSSSS	1 + 1 + 1 0+ 1 + 1 = 5	
	Total 8	

Metre (chandaḥ) Length, n	0	1	2	3	4	5	6	7	8	9
Fibonacci Sequence, F_{n+1}	1	1	2	3	5	8	13	21	34	55

Bharata

Short syllable (*laghu*) – 1 *mātrā*

Long syllable (*guru*) – 2 *mātrā*

No. of Beats, n	*1*	*2*	*3*	*4*	*5*
	L	G	LG	GG	LGG
		LL	GL	LLG	GLG
			LLL	LGL	LLLG
				GLL	GGL
				LLLL	LLGL
					LGLL
					GLLL
					LLLLL
Fibonacci Sequence	1	2	3	5	8

Piṅgala's Prastara (mātrā metres) – prastara (permutations)

n-Syllabic Metres and Variations

Meter Length, n	*1*	*2*	*3*	*4*
	G	GG	GGG	GGGG
	L	LG	LGG	LGGG
		GL	GLG	GLGG
		LL	LLG	LLGG
			GGL	GGLG
			LGL	LGLG
			GLL	GLLG
			LLL	LLLG
				GGGL
				LGGL
				GLGL
				LLGL
				GGLL
				LGLL
				GLLL
				LLLL
	1	4	8	16

Define grouping/clubbing pattern

1 – metre of four *laghu* or four *guru*

4 – metre of three *laghu* and one *guru* or one *laghu* and three *guru*

6 – metre of two *laghu* and two *guru*

Matrix Form of Pascal Triangle–Blaise Pascal (1623-62)

6-Row Pascal Triangle Merupastara – Piṅgala

Notes/ Syllables															*F*	*Piṅgala's pattern Variations*	*Līlāvatī Metre*
0							1								$(x+y)^0$	1	
1						1		1							$(x+y)^1$	2	*Ukta*
2					1		2		1						$(x+y)^2$	4	*Atyukta*
3				1		3		3		1					$(x+y)^3$	8	*Madhya*
4			1		4		6		4		1				$(x+y)^4$	16	*Pratiṣṭhā*
5		1		5		10		10		5		1			$(x+y)^5$	32	*Supratiṣṭha*
6	1		7		21		35		21		7		1		$(x+y)^6$	64	*Gāyatrī*

Matrix Form

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	2^n
0	1	0	0	0	0	0	0	1
1	1	1	0	0	0	0	0	2
2	1	2	1	1	0	0	0	4
3	1	3	3	1	0	0	0	8
4	1	4	6	4	1	0	0	16
5	1	5	10	10	5	1	0	32
6	1	6	15	20	15	6	1	64

Binomial coefficient

$$\binom{n}{r} = \begin{cases} \dfrac{n!}{(n-r)!\,r!}, o \le r \le n \\ o,\ r > n \end{cases}$$

is the coefficient of x^r in the expansion of $(1 + x)^n$.

Eventually, diagonal sums of Pascal triangle are Fibonacci sequence.

Further

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1},\ n,\ r \ge 2$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

Fibonacci numbers are generated thus:

$$F_1 = 1 = \binom{0}{0}$$

$$F_2 = 1 = \binom{1}{0}$$

$$F_3 = 1 + 1 = 2 = \binom{2}{0} + \binom{2}{1}$$

$$F_4 = 1 + 2 = 3 = \binom{3}{0} + \binom{2}{1}$$

$$F_5 = 1 + 3 + 1 = 5 = \binom{4}{0} + \binom{3}{1} + \binom{3}{2}$$

$$F_n = \binom{n-1}{0} + \binom{n-2}{0} + \binom{n-3}{2} + \ldots = \sum_{j=0}^{n} \binom{n-j-1}{j},\ j = 0,1,2....$$

FIBONACCI SEQUENCE IN GAṆITAKAUMUDĪ

Concept of Fibonacci numbers are more advanced in the *Gaṇitakaumudī*. Chapter 13 of it defines *sāmāsika-paṅkti* (additive sequence). Fibonacci numbers are particular case of this sequence.

Rule for the formation of *sāmāsika-paṅkti*:

> First keeping unity twice, write their sum ahead. Write ahead of that, the sum of numbers from the reverse order (and in) places equal to the greatest digit, write the sum of those (in available places). Numbers at places (equal to) one more than the sum of digits happen to be the *sāmāsika-paṅkti*. – Kak 2004

Let $v(q, r)$ be the r^{th} term of *sāmāsika-paṅkti* when the greatest digit is q. The rule implies $v(q, 1) = 1$ and $v(q, 2) = 1$. Let p stands for the number of places.

For

$$v(q, r) = \begin{cases} v(q, r-1) + v(q, r-2) + \ldots v(q, 2) + v(q, 1) 3 \le r \le q \\ v(q, r-1) + v(q, r-2) + \ldots v(q, r-q), \ q < r \end{cases}$$

$r = 1, 2, \ldots, n$. n is the sum of digits.

For $q = 2$, we obtain Fibonacci numbers.

Example (*Cow Problem*, *Gaṇitakaumudī*):

> A cow gives birth to a calf every year. The calves become young and they begin giving birth to calves when they are three years old. Tell me Oh learned man! the number of progeny produced during twenty years by one cow. – Kak 2004

Example: Piano (saptaka-octave)

> Fibonacci numbers and music are related. In music, an octave is an interval between two pitches, each of which is represented by the same musical note. The difference is that the frequency of the lower note is half that of the higher note. On the piano's keyboard, an octave consists of five black keys and eight white keys, totalling 13 keys. In addition, the black keys are divided into a group of two and a group of three keys. – Meinke 2011

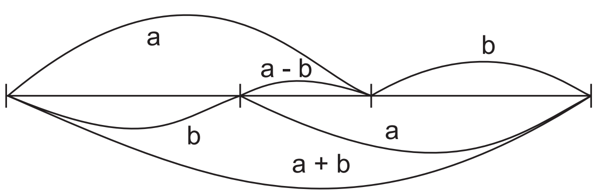
Piano's keyboard

Algebraic & Geometrical Structure:
Properties of Fibonacci Numbers

(Omotehinwa and Raman 2013; Rose 2014; Stakhov 2006)

Golden ratio (Fibonacci)

$$\frac{a+b}{b} = \frac{a}{b}$$



Writing $x = \frac{a}{b}$, we get $x = 1 + \frac{1}{x}$, i.e. $x^2 - x - 1 = 0$.
From which we find

$$\varphi = x = \frac{1+\sqrt{5}}{2} = 1.61803989$$

and

$$\psi = \frac{1}{\varphi} = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2} - 1 = \varphi - 1$$

Notice that $\varphi - \psi = 1$, $\varphi\psi = 1$

$$\varphi^2 = \varphi + 1$$

$$\varphi^3 = \varphi^2 + \varphi = 2\varphi + 1$$

$$\varphi^4 = \varphi^3 + \varphi^2 = 3\varphi + 2$$

$$\varphi^5 = \varphi^4 + \varphi^3 = 5\varphi + 3$$

Proceeding,

$$\varphi^n = F_n\varphi + F_{n-1},\ n = 1, 2, \ldots$$

$$\psi^n = F_n\psi + F_{n-1},\ n = 1, 2, \ldots$$

Subtracting we obtain, Binet's formula

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi},\ n = 1, 2, \ldots$$

Observe the asymptotic behaviour

$$\frac{F_{n+1}}{F_n} \to \varphi \text{ as } n \to \infty$$

$$F_{n+1} = \sum_{j=n-1}^{n} \binom{n-j}{j} = \sum_{j=1}^{n} \binom{n-j}{j-1}$$

$$= \sum_{j=1}^{n-1} \binom{n-j-1}{j} + \sum_{j=1}^{n-2} \binom{n-j-2}{j}$$

$$= F_n + F_{n-1.}$$

i.e.

$$F_n = F_{n-1} + F_{n-2},\ F_0 = F_1 = 1.$$

$$F_n = \sum_{k=1}^{n} \binom{n-k}{k},\ n > 1,\ n > 1$$

Let $F_n = r^n$. The equation will reduce to $r^2 - r - 1 = 0$. This gives φ, ψ.

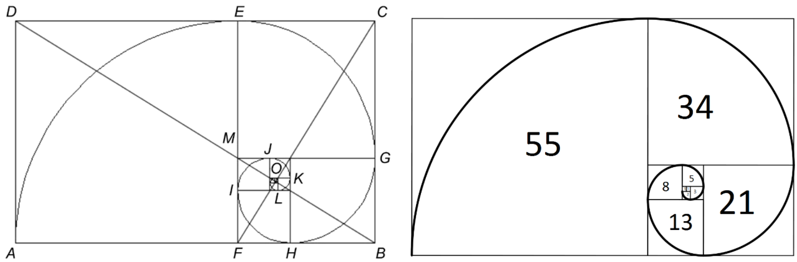
THEOREM (Vernon 2018)

If the ratio limit L of a Fibonacci type sequence exists, then it is a unique solution to the equation $x^n - x^{n-k} - 1 = 0$ in the interval $(1, \infty)$.

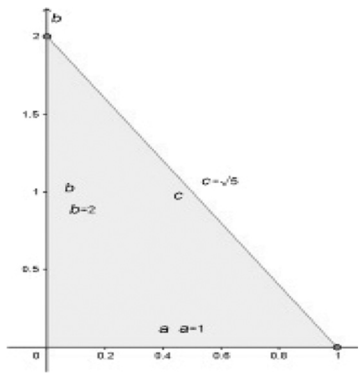
PYRAMID

Let y be half the base of square, h the height and s slant height of pyramid.

$$x^2 = h^2 + y^2$$



Golden spiral, or golden rectangle
(Overmars and Venkatraman 2018; Rose 2014)



Right-angled triangle representation of the golden ratio φ
(Overmars and Venkatraman 2018)

$$x^2 - y^2 = h^2 + xy$$

$$\begin{pmatrix} x \\ y \end{pmatrix}^2 - \begin{pmatrix} x \\ y \end{pmatrix} - 1 = 0$$

$$\varphi^2 - \varphi - 1 = 0$$

AREA OF SQUARE AND RECTANGLE

Area of square = area of rectangle

$$(x + y^2 = x(2x + y))$$

$$\left(\frac{x}{y}\right)^2 - \left(\frac{x}{y}\right) - 1 = 0$$

$$\varphi^2 - \varphi - 1 = 0$$

FIXED POINT ITERATION

$$x_{n+1} = \sqrt{1 + x_n},\ n = 1, 2, \ldots$$

or

$$x_{n+1} = 1 + \frac{1}{x_n},\ n = 1, 2, \ldots$$

NEWTON-RAPHSON METHOD (ORDER OF CONVERGENCE 2)

$$f(x) = x^2 - x - 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n)}$$

$$= \frac{x_n^2 + 1}{2x_{n-1} - 1}$$

OBSERVATIONS

1. Any two consecutive Fibonacci numbers are relatively prime.
2. For every two odd numbers, the next is an even number.
3. Sum of any ten consecutive Fibonacci numbers are always divisible by 11.
4. Fibonacci numbers in composite-number positions are always composite numbers, with the exception of the fourth Fibonacci number.
5. If n and m are positive integers, then $gcd(F_n, F_m) = F_{gcd(n,\, m)}$
6. F_n is divisible by F_m iff n is divisible by m.
7. Extended Fibonacci Numbers

Properties of Fibonacci Numbers

N	F_n	*Prime Factor*
1	1	
2	1	
3	2	
4	3	
5	5	
6	8	2^3
7	13	
8	21	3×7
9	34	2×7
10	55	5×11
11	89	
12	144	$2^4 \times 3^2$
13	233	
14	377	13×29
15	610	$2 \times 5 \times 61$
16	987	$3 \times 7 \times 47$
17	1597	
18	2584	$2^3 \times 17 \times 19$
19	4181	37×113
20	6765	$3 \times 5 \times 11 \times 41$
21	10946	$2 \times 13 \times 421$
22	17711	89×199
23	28657	
24	46386	$2^5 \times 3^2 \times 7 \times 13$
25	7502	$5^2 \times 3001$
50	12,586,269,025	

$$F_{-n} = (-1)^{n+1} F_n$$

and

$$F_n = \begin{cases} \dfrac{\varphi^n + \varphi^{-n}}{\sqrt{5}}, n = 2k+1 \\ \dfrac{\varphi^n - \varphi}{\sqrt{5}}, n = 2k \end{cases}$$

$$k = 0 \pm 1, \pm 2, ..$$

N	1	2	3	4	5	6	7	8	9	10	11
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	– 1	2	– 3	5	– 8	13	– 21	34	– 55

EXAMPLE: BEES AND RABBIT PROBLEMS
(Omotehinwa and Ramon 2013; Rose 2014; Scott and Marketes 2014)

Rabbit Problem

Growth pattern of the Fibonacci rabbit was first idealized by Fibonacci in his book *Liber Abaci* (1202).

Statement of Problem (Liu 2018)

The idea follows as:

> Rabbits never die; it takes one month for a pair of infant rabbits to become a pair of adults; an adult pair always gives birth to an infant pair; the system starts with one pair of adult rabbits. This gives rise to (Fibonacci sequence), where F_t is the number of adult pairs at month t, and the number of infant pairs at month t is F_{t-1}. So, the ratio of the number of adult pairs over the number of infant pairs goes to ϕ.

A man put a pair of rabbits (a male and a female) in a garden that was enclosed. How many pairs of rabbits can be produced from the original pair within twelve months, if it is assumed that every month each pair of rabbits produce another pair (a male and a female) in which they become productive in the second month and no death, no escape of the rabbits and all female rabbits must be reproduced during this period (year)? (Meinke 2011) (translation from *Liber Abaci* of Fibonacci).

The solution to this problem is Fibonacci numbers (sequence).

Explanation: Let us assume that a pair of rabbits (a male and a female) was born in January first. It will take a month before they can produce another pair of rabbits (a male and a female) which means no other pair except one in the first of February. Then, on first of March we have 2 pairs of rabbits. This will continue by having 3 pairs on the first April, 5 pairs on the first of May, 8 pairs on the June first and so on. The table below shows the total number of pairs in a year.

Total Number of Rabbit Pairs in a Year

Month	1 *Jan.*	2 *Feb.*	3 *Mar.*	4 *Apr.*	5 *May*	6 *June*	7 *July*	8 *Aug.*	9 *Sept.*	10 *Oct.*	11 *Nov.*	12 *Dec.*
Baby (Young)	1	0	1	1	2	3	5	8	13	21	34	55
Mature (Adult)	0	1	1	2	3	5	8	13	21	34	55	89
Total	1	1	2	3	5	8	13	21	34	55	89	144

Bees Problem

> We note that although the rabbit reproduction problem is not realistic, Fibonacci numbers fit perfectly to the reproduction ancestry of bees. Within a colony of bees, only the queen produces eggs. If these eggs are fertilized then female worker bees are produced. Male bees, which are called drones, are produced from unfertilized eggs. Female bees therefore have two parents; drones in contrast, have just one parent.
>
> –Scott and Marketos 2014

Further observations:

1. The male drone has one parent, a female.
2. He also has two grand-parents, since his mother had two parents, a male and a female.
3. He has three great-grandparents: his grandmother had two parents but his grandfather had only one and so forth.

Looking at the family tree of a male drone bee we note the following:

Comment

We observe that the ancestry of a worker or even a queen is simply a shifted Fibonacci sequence because of its connection to the ancestry of the bee drone. It is important to note that the number of ancestors at each generation n for (mammalian) sexual reproduction is simply 2^n. The ratio of two consecutive generations is asymptotically equal to 2 (Pāṇini sequence) whereas in the case of bees, it is asymptotically equal to the golden number 1.618.

Bee Family Tree

Generation	*Drone*	*Worker/Queen*
1	1	2
2	2	3
3	3	5
4	5	8
5	8	13
6	13	21
7	21	34
8	34	55
9	55	89
12	55	144

Thus the ancestry trees for bees and rabbits do not have the same mathematical complexity.

Fibonacci Code (Stakhov 2006)

Zeckendorf's system of writing numeral (Edou and Zerckendorf 1901-83)

$$N = a_n F_n + a_{n-1} F_{n-1} + \ldots + a_1 F_1,\ a_i \in \{0, 1\}$$

We write $N = a_n a_{n-1} \ldots a_1 F_1$

$$N = F_i + r,\ 0 \le r < F_{i-1} i = 2, 3, \ldots F_1 = F_2 = 1$$

$$F_i \le \text{N} < F_{i+1}$$

i.e. $0 \le N - F_i < F_{i+1} - F_i$

i.e. $0 \le r < F_{i-1}$

N	$F_5 = 8$	$F_4 = 5$	$F_3 = 3$	$F_2 = 2$	$F_1 = 1$	*Fibo Representation*
0	0	0	0	0	0	0
1	0	0	0	0	1	1
2	0	0	0	1	0	10
3	0	0	1	0	0	10
4 = 3 + 1	0	0	1	0	1	101
5	0	1	0	0	0	100
6 = 5 + 1	0	1	0	0	1	1,001
7 = 5 + 2	0	1	0	1	0	1,010
8	1	0	0	0	0	10,000

Fibonacci coding 11 = 8+ 2 + 1 = 1 × 8 + 0 × 5 + 0 × 3 + 1 × 2 + 1 × 1 = 10,011

Binary coding 11 = $2^3 + 2^1 + 2^0$ =1,011

Fibonacci code = Fibonacci encoded value + '1'.

Procedure

1. Find $M_i = \max F_i \leq N$. Note down the remainder
2. Put 1 in the i^{th} position of M_i.
3. Repeat the step 1. Repeat the process until we reach a remainder of zero.
4. Place 1 after the last naturally occurring one in the output.
5. We may put 1 in the Fibonacci code as 01, 11, 101, 1001, etc.

Level									
0	1	1	2	3	5	8	13	21	34
1	3	4	7	11	18	29	47	76	123
2	4	6	10	16	26	42	64	110	178
3	6	9	15	24	39	63	102	165	267
4	8	12	20	32	52	84	136	220	356

Range of Fibonacci number 0, 1, 1, 2, 3, 5, 8, 13, ... with $n = 5$ is from $F_0 = 0$ to $F_6 = F_5 = F_4 = 8 + 5 = 13$ is 13.

Modular Form

Let $F_0 = F_t \bmod p$, $F_1 = F_{t+1} \bmod p$, F_t is the t^{th} Fibonacci number. $F_n \bmod p$ form a periodic sequence, i.e. the sequence keeps repeating its values periodically.

F_n in Modular Form

F_p	Period, l_p	
7	16	0, 1, 1, 2, 3, 5, 1, 6, 0, 6, 6, 5, 1, 2, 6, 1
11	10	0, 1, 1, 2, 3, 5, 8, 2, 10, 1
13	28	0, 1, 1, 2, 3, 5, 8, 0, 8, 8, ..., 2, 12, 1
17	36	0, 1, 1, 2, 3, 5, 8, 0, 8, 13, 4, 0, 4,...,..., 2, 16, 1

Generalized Fibonacci Sequence

FIBONACCI SEQUENCE GENERATING FUNCTION
(AUSTIN ROCHFORD)

$$
\begin{aligned}
F(x) = \sum_{n=0}^{\infty} F_n x^n &= x + \sum_{n=2}^{\infty} F_n x^n \\
&= x + \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n \\
&= x + x\sum_{n=1}^{\infty} F_{n-1} x^{n-1} + \sum_{n=1}^{\infty} F_{n-2} x^{n-2} \\
&= x + xF(x) + x^2 F(x)
\end{aligned}
$$

$$\therefore F(x) = \sum_{n=1}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}.$$

Now,

$$
\begin{aligned}
F(x) &= \frac{x}{1 - x - x^2} = \frac{x}{(x + \varphi)(x + ø)} \\
&= \frac{1}{\sqrt{5}}\left(\frac{ø}{x + ø} - \frac{\varphi}{x + \varphi}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1 + x/ø} - \frac{1}{1 + x/\varphi}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \varphi x} - \frac{1}{1 - ø x}\right) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} (\varphi^n - ø^n) x^n \sum_{n=0}^{\infty} F_n (x) x^n
\end{aligned}
$$

GOPĀLA–HEMACANDRA SEQUENCE

$a, b, a + b, a + 2b, 2a + 3b, \ldots$

$a = 1, b = 1$ corresponds to Fibonacci numbers

$a = 2, b = 1$ corresponds to Lucas numbers

EXTENDED FIBONACCI NUMBERS

$$F_n = \sum_{i=1}^{K} a_i F_{n-i}.$$

TRIBONACCI NUMBERS

$$F_{n+1} = F_n + F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 0, F_2 = 1.$$

GENERALIZATION OF MOUNT MERU (KAK 2004)

$$F_{n+1} = F_n + F_{n-1} + F_{n-2}, F_0 = F_1 = 0, F_2 = 1$$

Metre (*chandaḥ*) Length, n	0	1	2	3	4	5	6	7	8	9
Fibonacci Sequence, F_n	0	0	1	1	2	4	7	13	24	44

Triplicate Meru

1						1					
2					1	1	1				
3				1	2	3	2	1			
4			1	3	6	7	6	3	1		
5		1	4	10	16	19	16	10	4	1	
6	1	5	15	30	45	51	45	30	15	5	1

STATISTICAL APPLICATION OF GENERALIZED FIBONACCI SEQUENCE (Cooper 1984)

A fair coin is tossed repeatedly until n consecutive heads are obtained. What is the expected number of tosses e_n to conclude the experiment?

GENERALIZATION OF GOLDEN SECTION (GOLDEN P-SECTION) (Agaian and Gill III 2017; Agaian 2009)

$\frac{CB}{AC} = \left(\frac{AB}{CB}\right)^p$, p is a positive integer. $CB > AC$, $AB = AC + CB$.

This implies $x^{p+1} - x^p - 1 = 0$, $x = AB/CB$. Positive root is called golden p-ratio.

Generalized Fibonacci numbers

$$F_n^{(p)} = F_{n-1}^{(p)} + F_{n-p-1}^{(p)}, n > p+1$$
$$F_1^{(p)} = F_2^{(p)} = \ldots = F_{n+1}^{(p)}$$

Further, $x = \lim_{n\to\infty} \frac{F_n^{(p)}}{F_n^{(p)}}$, $p=0,1,2,\ldots$

$$F_n^{(p)} = \begin{cases} 0, n \le 0 \\ 1, 0 < n \le p+1 \\ F_{n-1}^{(p)} - F_{n-p-1}^{(p)}, n > p+1 \end{cases}$$

p	*Fibonacci Numbers*								
0	0	1	2	4	6	8	16	32	64
1	0	1	1	2	3	5	8	13	
2	0	1	1	1	2	3	4	6	9
3	0	1	1	1	1	2	3	4	
4	0	1	1	1	1	1	2	3	4

Application: Generalized golden ratio is generally applied for forecasting financial time series analysis (simulation). This includes correlation analysis, moving averaging models, logistic regression, artificial neural networks, support vector machines and decision tree analysis (Agaian and Gill III 2017; Agaian 2009).

Fibonacci Polynomials

Fibonacci polynomials are obtained from generalized or weighted Fibonacci sequence as follows (Araghi and Noeiaghdam 2017; Bashi and Yelcinbas 2016; Kurt and Sezer 2013; Mirzaee and Hoseini 2013):

Let k be an integer. Then k-Fibonacci sequence is defined by

$$F_{n+1} = kF_n + F_{n+1}, F_1 = F_0 = 1$$

For

$$k = 1, F_{n+1} = F_n + F_{n+1} \text{ (Fibonacci sequence)}$$

If $k = x$ is a real variable, then

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x), F_0(x) = F_0 = 0, F_1(x) = F_1 = 1$$

$$F_{n+1}(x) = \begin{cases} 1, & n = 0 \\ x, & n = 1 \\ xF_n(x) + F_{n-1}(x), & n > 1 \end{cases}$$

$[n/2]$ stands for greatest integer not exceeding $n/2$. This is equal to $(n - 1)/2$ if n is even and $n/2$ if n is odd.

$F_1(x) = 1$

$F_2(x) = x$

$F_3(x) = x^2 + 1$

$F_4(x) = x^3 + 2x$

$F_5(x) = x^4 + 3x^2 + 1$

$$F_{n-1}(x) = \sum_{i=1}^{|n/2|} \binom{n-i}{i} x^{n-2i}, n > 0.$$

The polynomials so obtained are used in solving differential, integral and difference equations wherein solutions are expressed in matrix equivalent of linear combinations of Fibonacci sequence.

$$y(x) = \sum_{r=1}^{N} a_r F_r(x) = F(x)A$$

where

$$F = [F_1(x), ..., F_N(x)], A = [a_1, ..., a_N]^T.$$

For further procedural details refer to of pro Equations (Koc et al. 2014; Mirzaee and Hoseini 2013).

Fibonacci Sequence in Biology and Medicine

ENERGY SOURCES

- Carbohydrates (starch, cellulose, glucose)
- Proteins (*daal*-cereals, meat, eggs)
- Lipid (ghee/oil, fatty acids)
- Nucleic acid (DNA, RNA)

DNA/RNA are combinations of sugar, phosphates and nucleic (nitrogenous) bases called nucleotides. Nucleic bases are divided into purine (adenine-*A*, guanine-*G*) and pyrimidine (thymine-*T*, cytosine-*C*, uracil-*U*).

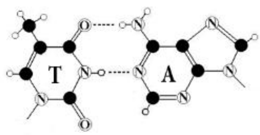
Basic element of DNA are the sequence (polymer) of four nitrogenous bases: *A*, *G* and *C*, *T* while RNA is made up of *A*, *G* and *C*, *U*.

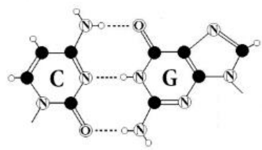
> In human field, Dress et al. proposed that the growth pattern of repetitive DNAs is analogous to the pattern described by the Fibonacci process (repetitive DNAs are those built from a basic short DNA sequence that is repeated many times, often referred to as "junk DNA" and accounted for a large fraction of the whole human genome).
>
> – Liu and Sumtler 2018

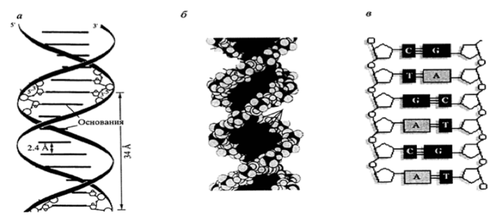
MATRIX REPRESENTATION OF DNA MOLECULES

(Hu and Pitoulchov 2017; Koblyakov et al. 2011)

The pairs of complementary molecules $A - T$ and $C - G$ of DNA are respectively linked by 2 and 3 hydrogen bonds.







RELATION BETWEEN GENETIC MATRIX AND GOLDEN SECTION

Genetic matrix ([3, 2; 2, 3]^n) = [φ, φ^{-1}, φ^{-1}, φ]^n

$$[CA;TG]^{(1)} = \begin{bmatrix} C & A \\ T & G \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}^{1/2} = \begin{bmatrix} \varphi & \varphi^{-1} \\ \varphi^{-1} & \varphi \end{bmatrix}$$

$$[CA;TG]^{(2)} = \begin{bmatrix} CC & CA & AC & AA \\ CT & CG & AT & AG \\ TC & TA & GC & GA \\ TT & TG & GT & GG \end{bmatrix}, \begin{bmatrix} 9 & 6 & 6 & 4 \\ 6 & 9 & 4 & 6 \\ 6 & 4 & 9 & 6 \\ 4 & 6 & 6 & 9 \end{bmatrix}^{1/2} = \begin{bmatrix} \varphi^2 & \varphi^0 & \varphi^0 & \varphi^{-2} \\ \varphi^0 & \varphi^2 & \varphi^{-2} & \varphi^0 \\ \varphi^0 & \varphi^{-2} & \varphi^2 & \varphi^0 \\ \varphi^{-2} & \varphi^0 & \varphi^0 & \varphi^2 \end{bmatrix}$$

	0	1
0	*C*	*A*
1	*T*	*G*

	00	01	10	11
00	*CC*	*CA*	*AC*	*AA*
01	*CT*	*CG*	*AT*	*AG*
10	*TC*	*TA*	*GC*	*GA*
11	*TT*	*TG*	*GT*	*GG*

$[CA; TG]^{(1)}$ contains two numbers 3, 2; ratio 3/2.

$[CA; TG]^{(2)}$ contains three numbers 9, 6, 4; quint ratio each 3/2.

$[CA; TG]^{(3)}$ contains four numbers 27, 18, 12, 8; quint ratio each 3/2.

The concept of Fibonacci sequence is used to study potential number of fatty acids and maturation problem of cell division process.

References

Agaian, Sarkis and John T. Gill III, 2017, "The Extended Golden Section and Time Series Analysis", *Frontiers in Signal Processing*, 1: 67-80.

Agaian, Sarkis, 2009, "Generalized Fibonacci Number and Applications", *Proceedings of IEEE International Conference on Systems, Man and Cybernatics*, pp. 3484-88.

Araghi, Mohammad A.F. and Samad Noeiaghdam, 2017, "Fibonacci-Regularization Method for Solving Cauchy Integral Equations of the First Kind", *Ain Shams Engineering Journal*, 8: 365-69.

Bashi, A.K. and Sallih Yalcinbas, 2016, "Fibonacci Collocation Method with a Residual Error Function to Solve Linear Volterra Intego Differential Equation", *New Trends in Mathematical Sciences*, 4: 1-14.

Boman, B.M., Thien-Nam Dinh, Keith Decker, Brooks Emerick, Christopher Raymond and Gilberto Schleiniger, 2017, "Why Do Fibonacci Numbers Appear in Patterns of Growth in Nature?: A Model for Tissue Renewal Based on Asymmetric Cell Division", *Fibonacci Quarterly*, 55: 30-41.

Cooper, Curtis, 1984, "Application of a Generalized Fibonacci Sequence", *The College Mathematics Journal*, 15: 145-46.

Hall, Rachel, 2008, "Math for Poets and Drummers", *Math Horizons*, 15(3): 10-24.

Hu, Z.B. and S.V. Pitoulchov, 2017, "Generalized Crystellography, the Genetic System and Biological Esthetics", *Struc. Chem.* 28: 239-47.

Kak, Subhash, 2004, "The Golden Mean and the Physics of Aesthetics", *Archive of Physics*, 1-9.

Koblyakov, A., S. Petoukhov and I. Stepanian, 2011, "The Genetic Code, the Golden Section and Genetic Music", PPT, petoukhov.com/wp-content/uploads/2011/05/sciforum-003889-from-PETOUKHOV.pdf.

Koc, AyGe Betül, Musa Çakmak and AydJn Kurnaz, 2014, "A Matrix Method Based on the Fibonacci Polynomials to the Generalized Pantograph Equations with Functional Arguments", *Advances in Mathematical Physics*, 1-5.

Kuhapatanakul, Kantaphon, 2016, "The Fibonacci *p*-numbers and Pascal's Triangle", *Cogent Mathematics*, 3: 1264-76.

Kurt, Ayse, Salih Yalçinbas and Mehmet Sezer, 2013, "FIBONACCI Collocation Method for Solving Linear Differential: Difference Equations", *Mathematical and Computational Applications*, 18: 448-58.

Liu, Yu and David J.T. Sumpter, 2018, "Is the Golden Ratio a Universal Constant for Self-replication?", *PLOS One*, 1-18.

Meinke, A.M., 2011, "Fibonacci Numbers and Associated Matrices", Master of Science Thesis, Kent State University.

Mirzaee, Farshid and Seyede Fatemeh Hoseini, 2013, "Solving Singularly Perturbed Differential – Difference Equations Arising in Science and Engineering with Fibonacci Polynomials", *Results in Physics*, 3: 134-41.

Mishra, Vinod, 2002, "Combinatorics and the (So-called) Binomial Theorem", *Studies in History of Medicine and Science* 18: 39-121.

Omotehinwa, T.O. and S.O. Ramon, 2013, "Fibonacci Numbers and Golden Ratio in Mathematics and Science", *International Journal of Computer and Information Technology*, 2: 630-36.

Overmars, Anthony and Sitalakshmi Venkatraman, 2018, "An Efficient Golden Ratio Method for Secure Cryptographic Applications", *Mathematical and Computational Applications,* 23: 1-12.

Rose, N.J., 2014, *The Golden Mean and Fibonacci Numbers*, studylib.net/doc/7967929/.

Schuster, Stefan, 2017, "Maximilian Fichtner and Severin Sasso, Use of Fibonacci Numbers in Lipidomics: Enumerating Various Classes of Fatty Acids", *Scientific Reports*, 7: 39821.

Scott, T.C. and P. Marketos, 2014, "On the Origin of the Fibonacci Sequence", *MacTutor History of Mathematics*, 23 March, mathshistory.st-andrews.ac.uk/Publications/fibonacci.pdf.

Sen, S.K. and R.P. Agarwal, 2008, "Golden Ratio in Science, as Random Sequence Source, Its Computation and Beyond", *Computers and Mathematics with Applications*, 56: 469-98.

Shah, Jayant, 2008, A *History of Pingala's Combinatorics*, Boston: Northeastern University.

Shah, Saloni, 2010, *An Exploration of the Relationship between Mathematics and Music*, Manchester University, PPT.

Shannon, A.G., Irina Klamka and Robert van Gend, 2018, "Generalized Fibonacci Numbers and Music, *Journal of Advances in Mathematics*, 14: 7564-75.

Sheshadri, C.S. (ed.), 2000, *Studies in the History of Indian Mathematics,* New Delhi: Hindustan Book Agency (Raja Sridharan, R. Sridharan and MD Srinivas, Combinatorial Methods in Indian Music).

Singh, Parmanand, 1985, "The So-called Fibonacci Numbers in Ancient and Medieval India", *Historia Mathematica*, 12: 229-44.

Stakhov, Alexey, 2006, "Fundamentals of a New Kind of Mathematics Based on the Golden Section", *Chaos, Solitons and Fractals*, 27: 1124-46.

Tiwai, Sankalp and Anurag Gupta, 2017, "Effects of Air Loading on the Acoustics of an Indian Musical Drum", *J. Acoust, Soc. America*, 141: 2611-21.

Vernon, R.P., 2018, "Relationships between Fibonacci-Type Sequences and Golden-Type Ratios", *Notes on Number Theory and Discrete Mathematics*, 24: 85-89.

12

Karaṇī (Surds)

R. Padmapriya

बहुभिर्विप्रलापैः किं त्रैलोक्ये सचराचरे।
यत्किञ्चिद्वस्तु तत्सर्वं गणितेन विना न हि।।

Whatever there is in all the three worlds which are possessed of moving and non-moving being all that indeed cannot exist without *gaṇita.* – *Gaṇitasāra-saṁgraha* I.16

GAṆITAŚĀSTRA has always occupied a position of high ranking among the various Śāstras. This is seen right from the Vedic period. The importance of learning *gaṇita* for learning Vedas as well as for performing sacrifices and rituals is clearly evident from the texts of the Vedic period.

Among the Vedāṅgas, Kalpa occupies a special place. Kalpa provides all necessary details under different heads, viz. Śrautasūtras, Gṛhyasūtras, Dharmasūtras and Śulbasūtras. Among these, Śulbasūtras deals with the rules and measurements for constructing the fire alter. Śulbasūtras can possibly be treated as the earliest mathematical texts in India.

The *vedī*s were constructed in different shapes, such as *rathacakraciti* (circle), *śyenaciti* (a bird-shape), *ubhayata prauga* (rhombus shape) and *kūrmaciti* (tortoise shape). For these constructions, they needed the knowledge of geometry.

The Śulbasūtras like Baudhāyana, Āpastamba, Mānava and

Kātyāyana have given the rules for constructing altars. One can learn most of the geometrical rules from the Śulbasūtras. They also give rules for arrangement as well as measurements of the bricks. Baudhāyana was the first to give all the geometrical rules. He gave the rules for finding approximate value of and the theorem of square (popularly known as Pythagorean Theorem). Though the development of Gaṇitaśāstra is found in Vedic period from fifth century CE onwards the other mathematicians such as Brahmagupta, Varāhamihira, Mahāvīra, Śrīdhara, Śrīpati and Bhāskara II enriched Gaṇitaśāstra.

The Word Karaṇī in Vedic Period

In the Śulbasūtras, the ancient work on geometry, the *sulbakāra*s used the word *akṣāṇayārajju* to designate the diagonal of a square or rectangle, whereas the length and breadth were the *tiryaṅmāṇi* and *pārśvamāṇi*:

> दीर्घचतुरस्रस्याक्ष्णयारज्जुः पार्श्वमानी तिर्यङ्मानी च यत्पृथग्भूते कुरुतस्तदुभयं करोति।।
> – *Baudhāyana Śūlbasūtra* I.48

But the word *akṣṇayārajju* disappeared after a while and the word *karṇa* was substituted for the word "diagonal".

Since the *śulbakāra*s used the rope to cut a figure along its diagonal, the word *karṇa* is used as a modifier of the word *rajju*, which means a rope. Since the word *rajju* is a feminine noun, the adjective *karṇa* (making) takes the feminine form as *karaṇī.* The word *karaṇī* occurs very frequently in the Śulbasūtras. Baudhāyana treats the word *karaṇī* as side of the square formed on the diagonal produced by *rajju.* Āpastamba also uses the word *dvikaraṇī* in the sense of a measurement by a rope. Kātyāyana treats the word *karaṇī* as to mean a rope.

> करणी तत्करणी तिर्यङ्मानी पार्श्वमान्यक्ष्णया चेति रज्जवः।
> – *Kātyāyanā Śulbasūtra* II.3

The terms *karaṇī, tatkaraṇī, tiryaṇmānī, pārśvamānī* and *akṣṇayā* denote chords (measuring the side of a square or rectangle).

Āryabhaṭa I uses the word *varga kṛti* for square power and *mūla* for square root. He never uses the word *karaṇī* in either sense. In his work *Āryabhaṭīya*, he gives the rules for the construction of circle, triangle and quadrilateral, where the word *karṇa* is used to denote hypotenuse of triangle and diagonal of quadrilateral.

वृत्तं भ्रमेण साध्यं त्रिभुजं च चतुर्भुजं च कर्णाभ्याम्।
साध्या जलेन समभूरध ऊर्ध्वं लम्बकेनैव।।
– *Āryabhaṭīya* II.13

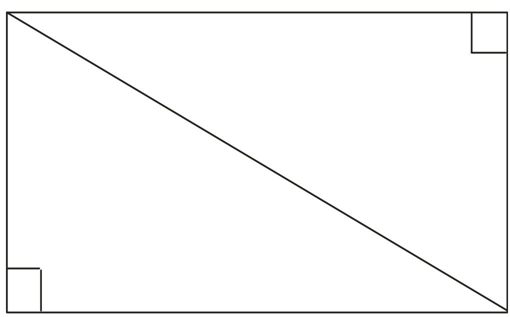
A circle should be constructed by means of a pair of compasses while a triangle and a quadrilateral are constructed by means of two hypotenuse.

He also gives the theorem of square of hypotenuses:

यश्चैव भुजावर्गः कोटीवर्गश्च कर्णवर्गः सः।
– *Āryabhaṭīya* II.17

$(bhuj\bar{a})^2 + (ko\d{t}\bar{\imath})^2 = (kar\d{n}a)^2$.

Here, the term *bhujā* means base of a rectangle or square, *koṭī* means altitude and *karṇa* means hypotenuse and diagonal. This is represented in the following figure:



Later on, the theorem was called *bhujā – koṭi – karṇa – nyāya.*

This can be compared with the Pythagorean Theorem of 500 CE as follows:

> In a right-angled triangle, where c is the hypotenuse and a and b are the other two sides, it can be stated that: $a^2 + b^2 = c^2$.

The *karaṇī* is so called because it makes (*karoti*) the equation of hypotenuse – c and sides a and b, i.e. $a^2 + b^2 = c^2$. Here, the word *karaṇī* is taken from the root *kṛ* (*kar*) to do.

The *Amarakośa,* the ancient Sanskrit text on lexicography, gives the synonyms of *karaṇī* as *śrotram, śrutiḥ, śravaṇam* and *śravaḥ.*

In his *Nirukta,* Yāska derives the word *karṇa,* from the root *kṛt,* to cut. The word *karṇa* will take the meaning "a line dividing a figure". Since the line diagonal cuts the figure of rectangle or square, a diagonal can also be designated as *karṇa.* The Greek root, *krino* which means "to separate", "put asunder", is similar to *karṇa* and so supports the etymological derivation of Yāska.

From the above statement, we are able to understand that the word *karaṇī* was in usage from Vedic period, even before the period of Bhāskara I and was used to denote the diagonal and hypotenuse, and that its usage is more in works of geometry.

Varga, karaṇī, kṛtir, vargaṇa, yavakaraṇam are synonyms of *karaṇī.* When a number takes the quality of being *karaṇī,* Bhāskara I calls it *karaṇītvam.* He employs it in this sense while explaining the volume of a pyramid in the *Āryabhaṭīya Bhāṣya*:

अर्धमित्यत्र करणित्वाद् द्वयोः करणीभिश्चतुभिर्भागो ह्रियते।

Here, when half takes the quality of *karaṇī,* it is mentioned as *karaṇītva,* whereas the term *karaṇī* refers to the surds.

Another line in the *Āryabhaṭīya Bhāṣya*, while explaining the volume of a sphere, clearly shows:

तत्पुनः क्षेत्रपलं मूलक्रियमाणं करणित्वं प्रतिपद्यत, यस्मात्करणीनां मूल (मपेक्षितम्)।

Karaṇī is a number whose square root is to be taken. But the area, when its square root is being taken, obtains the state of being *karaṇī* because the square root is required of *karaṇī.*[1]

Śrīpati, in his astronomical treatise, the *Siddhāntaśekhara* defines the term thus:

ग्राह्यं न मूलं खलु यस्य राशेस्तस्य प्रतिष्टं करणीति नाम।

– XIV.7ab

The name *karaṇī* has been given to a number whose square root

[1] *ABB* II.7, Eng. tr. Hayashi, p. 61.

should be obtained, but speaking exactly does not exist as an integer.

Brahmagupta and Bhāskara II also use the term in the same sense, although they do not give its definition.[2]

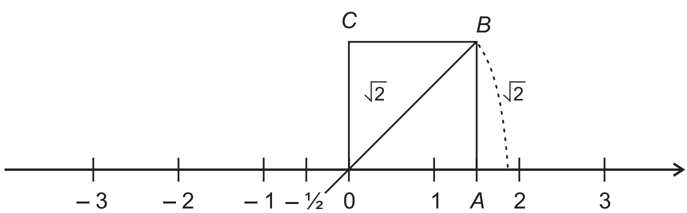
Mahāvīra, in his *Gaṇitasāra-saṁgraha* uses the word *karaṇī* with a short vowel (*karaṇī*) when he gives the rule for the addition and subtraction of the surd.[3]

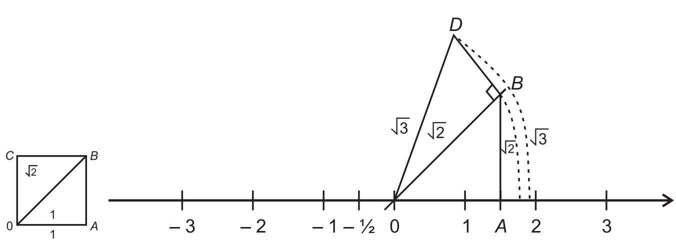
Bhāskara II in his *Līlāvatī* uses the word *karṇa* to denote hypotenuse and in *Bījagaṇita* he mentions surd numbers as *karaṇī.*

Surds in Modern Mathematics

In modern mathematics, surd is defined thus: "Surds are irrational numbers. They are non-terminating, non-repeating decimals."

Surd is a number which cannot be perfectly evaluated but which can be measured accurately. These numbers can be located on the number line. Representation of √2 and √3 is as follows:





Our *śulbakāra*s also express the same rule. *Dvikaraṇī* √2, *trikaraṇī* √3 cannot be calculated accurately. They give a method to find the approximate value of √2. But they measured the exact value

---

[2] *BrSpSi* XVIII.38-40, explained in chapter 3.

[3] *GSS, kṣetra* 88-89 explained in chapter 2.

of $\sqrt{2}$, $\sqrt{3}$ by measurement. This achievement in this field without any modern sophisticated tools is remarkable. Also in modern mathematics surds are generally known to be $\sqrt[n]{x}$ (where x cannot be written in the form y^n (where y is a whole number). But the word *karaṇī* in Gaṇitaśāstra mostly denoted the diagonal of a rectangle or square or a hypotenuse of a right angle triangle. Since the diagonal can only be a square root and not cube root, fourth root, or fifth root, the term *karaṇī* mostly denotes the square root of a non-perfect square number, which is called a surd. From the above study it can be conclude that the term *karaṇī* appears to match with our modern mathematical term "surd".

Karaṇī in Kṣetragaṇitam

When Bhāskara I gives an introduction to *gaṇitapāda* in his *Āryabhaṭīya Bhāṣya*, he starts thus:

> गणितं द्विप्रकारम् – राशि गणितम क्षेत्र गणितम्। अनुपातकुट्टाकारादयो गणित–विशेषाः राशिगणिते–अभिहिताः, श्रेढीच्छायादयः क्षेत्रगणिते। तदेवं राश्याश्रितं क्षेत्राश्रितं वा अशेषं गणितम् यदेतत् करणी–परिकर्म तत् क्षेत्रगणित एव। यद्यप्यन्यत्र करणीपरिकर्म, तथापि तस्य न कर्णभुजाकोटि प्रतिपादकत्व–मिति न दोषः।

Thus we understand that our ancient mathematicians classified *gaṇita* under two heads: *Rāśi-gaṇita* (symbolical mathematics) and *kṣetra-gaṇita* (geometrical mathematics). Topics like algebra fall under *rāśi-gaṇita*, while others like series problems on shadow fall under the *kṣetra-gaṇita*. The operations of surds (*karaṇī-parikarma*), though it formed part of algebra (*kuṭṭaka*), was essentially a part of geometry (*kṣetra-gaṇita*), for its main function was to establish relations between the hypotenuse, the base and the upright. The operations of *karaṇī* are also present in other chapters like arithmetic and algebra where the relation between hypotenuse and base is not mentioned. Early studies on *karaṇī* are found in geometrical works like the Śulbasūtras. This is because they deal with measurements of areas and lengths of lines and sides.

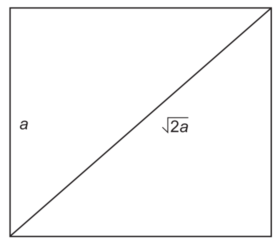
Method of constructing a square leads to the origin of *karaṇī* (surd number) in Śulbasūtras.

The Origin of Dvikaraṇī, Trikaraṇī, Tṛtīyakaraṇī

Baudhāyana explains *karaṇī* thus:

समचुतरस्रस्याक्ष्णयारज्जुर्द्विस्तावतीं भूमिं करोति।।

A square constructed on the diagonal of a square produces double the area of square.



In a square $ABCD$

$AC = 2BC^2$

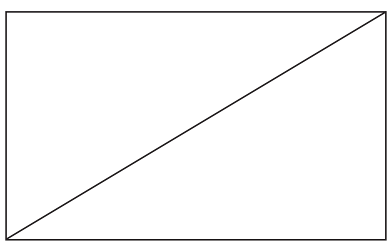
$AC^2 = 2AB^2$ $(AB = BC)$

$AC = \sqrt{2}AB. = \sqrt{2}a$ where AC is *dvikaraṇī* of the measure AB.

Then Baudhāyana gives rule for *trikaraṇī*:

प्रमाणं तिर्यग् द्विकरण्यायामस्याक्ष्णयारज्जुस्त्रिकरणी।।

Then again the measure of the diagonal of a rectangle, having sides a and $\sqrt{2}a$ is $\sqrt{3}a$, for $a^2 + (\sqrt{2}a)^2 = 3a^2 = (\sqrt{3}a)^2$ $\sqrt{3}a$ is known as *trikaraṇī*.



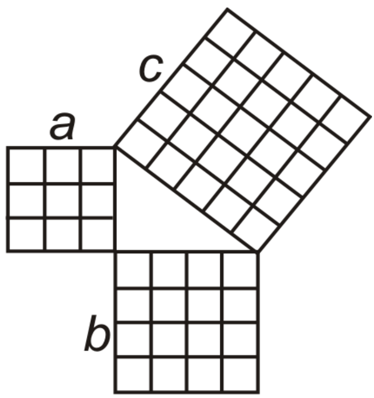
The knowledge of *dvikaraṇī* and *trikaraṇī* discussed by *śulbakāra*s led in a way to the theorem of square on a diagonal.

In modern mathematics we call it as Pythagorean Theorem. It seems that it was known in India before Pythagorus gave it to the world. In Śulbasūtras the actual theorem is in regard to a rectangle and not triangle. They considered right angle triangle as a part of rectangle and square. The Śulbasūtras explicitly did not give any name for the theorem of square. But our ancient Indian mathematicians called it as *bhujā – koṭi – karṇa – nyāya*.

The Origin of Bhujā–Koṭi–Karṇa–Nyāya

Baudhāyana explains the theorem of square thus:

दीर्घचतुरस्रस्याक्ष्णयारज्जुः पार्म्मानी तिर्यङ्मानी च यत्पृथग्भूते कुरुतस्तदुभयं करोति।।



The square *b* is constructed on the length (*tiryaṅmānī*) of the rectangle (*dīrghacaturasram*) while square *a* is constructed on the breadth (*pārśvamānī*) of the rectangle and square *c* is constructed on the diagonal (*akṣṇayārajju*) of the rectangle area of square *a* + area of square *b* = area of square *c*

$a^2 + b^2 = c^2$.

A proof of *bhujā–koṭi–karṇa–nyāya* is given by Bhāskara II in his *Bījagaṇitam*[4] in the form of an example. In this example, he suggests the method to find the hypotenuse of a right angle triangle whose other sides are given:

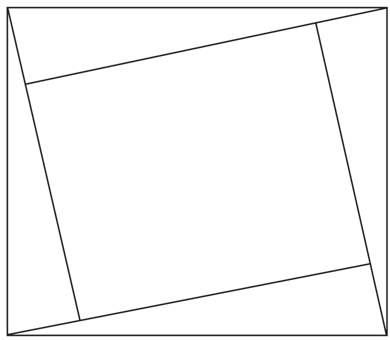
क्षेत्रे तिथि नखैः तुल्ये दोः कोटी तत्र का श्रुतिः।
उपपत्ति यः रूढस्य गणितस्यास्य कथ्यताम्।।

Here, the words *nakhaiḥ* and *tithi* denote the number 20 and 15 respectively. *Śrutiḥ* denotes *karṇa* (hypotenuse).

[4] Commentary on the *Bījagaṇitam* by Sudhakaradvivedi.

> Say what is the hypotenuse in a plane figure in which the side and upright are equal to fifteen and twenty? And show the demonstration of the received mode of computation.

The answer is 25.



Here, the answer is found by the construction of four right-angle triangles. The four right-angle triangles are arranged in such a way that their hypotenuses form a square. In the process, another interior square is formed. The difference between the upright and the base is the length of its side. From this, we can know that area of the square is equal to the area of four right-angle triangles with the area of the interior quadrilateral. Knowing the area of the square we can easily find the side of square which is the hypotenuse of the right-angle tringle.

The Value of $\sqrt{2}$ Given by Śulbakāras

The value is:

$$\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}=\frac{577}{408}=1.414215.$$

In modern mathematics, the value of $\sqrt{2} = 1.4142135$. The *śulbakāra*s attained a remarkable degree of accuracy in calculating an approximate value of $\sqrt{2}$ which is similar to the modern value.

The rule given by Baudhāyana is:

प्रमाणं तृतीयेन वर्धयेत्तच्च चतुर्थेनात्मचतुस्त्रिंशोनेन।। सविशेष।।

– *Baudhāyana Śulbasūtra* I.62

The measure is to be increased by its third and this again by its own fourth less the thirty-fourth part of that fourth. This is the value of the diagonal of a square.

Baudhāyana says that this value is approximate only. This is understood by the term *viśeṣa.* Thus, the *śulbakāra*s recognized the irrationality of √2.

Constructional geometry in Śulbasūtras is the origin of arithmetical and algebraic operations on surds. Among all the ancient Indian mathematicians, Bhāskara II was one of the very few authors who dealt elaborately with *karaṇī.* In his algebraical work *Bījagaṇitam* he deals with the operations on *karaṇī.*

Bhāskara II explains the process of addition, subtraction, multiplication, division, squaring and square root of surds.

The term *avyakta* means unmanifested thing. The six operations of surds are dealt elaborately in *avyakta-gaṇita.* In the above operations, the answers are only approximate, i.e. not manifested clearly. Surds are the numbers which do not have perfect square root values. Bhāskara's commentator Kṛṣṇadaivajña in his *Bījapallavam* mentions thus:

> संज्ञा तु करणीराशावेतस्य गणितस्यावश्यकत्वाद्द्रष्टव्या। तत्र यस्य राशेर्मूलेऽपेक्षिते निरग्रं मूलं न संभवति स करणी। न त्वमलदराशिमात्रम्।
>
> – Karaṇī Ṣaḍvidham chapter

Conclusion

The various definitions of *karaṇī* given by the ancient Indian mathematicians evolved so that they could gradually differentiate between the *varga* or *pada* (which refers to the square root of perfect squares) and *karaṇī* (which refers to the square root of the usage of the *karaṇī* such as √2, √3 and 1/√3 seem to have made it easy for the *śulbakāra*'s calculations. This is seen from the fact that they converted all big surds (which arose as a result of calculations) into these three *karaṇī* non-perfect squares).

The present study of surds plays a very important role in the astronomical field. This is because surds feature in a number of astronomical research studies like calculations of position of the sun, the planets and other heavenly bodies, the phenomenon of eclipse and many other areas of astronomy.

References

Amarakośa with south Indian comms (3 vols), ed. A.A. Ramanathan, Chennai: Adyar Library and Research Centre, 1978.

Amma, T.A. Saraswati, 1979, *Geometry in Ancient and Medieval India,* Delhi: Motilal Banarsidass.

Āpastamba Śulbasūtras with Sanskrit comm. by Dwaraka Nath Yajvana, ed. Satyaprakash and Ramswaroop Sharma, Delhi, 1968.

Āryabhaṭīya with comm. of Bhāskara I and Someśvara, ed. K.S. Shukla, Delhi: Indian National Science Academy, 1976.

Baudhāyana Śulbasūtras with comm. by D.N. Yajvana, ed. Satyaprakash and Ramswaroop Sharma, The Research Institute of Ancient Scientific Studies, Delhi, 1968.

Bījagaṇitam with *Subodhinī* comm. of Sri Jivanatha Jha, tr. Pt. Shri Acyutananta Jha, Varanasi: Chowkamba Samskrta Sansthana, 2002.

Gaṇitatilaka of Śrīpati with the comm. of Simhatilaka Sufi, critically ed. H.R. Kapadia, Baroda: Oriental Institute, 1937.

Gaṇitasāra-Saṁgraha of Mahāvīrācārya, ed. M. Rangacarya with Eng. tr. and notes, New Delhi: Cosmo Publications, 2011.

Hayashi, Takao (ed.), 1995, *The Bakshali Manuscript: An Ancient Indian Mathematical Treatise,* Groningen: Egbert Forsten.

Kātyāyana Śulbasūtra, ed. Khadilkar, Poona: Vaidika Samsodhana Mandala, 2003.

Līlavatī of Bhāskarācārya with *Tattvacandrikā* by Ramchandra Pandey, Varanasi: Krishnadas Academy, 1993.

Līlavatī of Bhāskarācārya, tr. H.T. Colebrooke, Asian Educational Services, 1993.

Sastri, T.V. Radhakrishna (ed.), 1958, *Bījapallavam,* by Tanjore Sarasvati Mahal Series, No. 78, Tanjore.

The Śulbasūtras, Text and Eng. tr. with comm. by S.N. Sen and A.K. Bag, New Delhi: Indian National Science Academy, 1993.

13

Square Roots of Expressions in Quadratic Surds as per Bhāskarācārya

S.A. Katre
Shailaja Katre
Mugdha Gadgil

Abstract: Bhāskarācārya in his *Bījagaṇita* talks about square roots of expressions in quadratic surds where more than one *karaṇī* (surd, i.e. a square root of a positive integer which is not a perfect square) may be present. Bhāskarācārya gives a method of finding a square root when the expression is a sum of an integer and one surd. This method depends only on finding the square of an integer and a square root of a perfect square. Algorithms for squaring and root finding have already been discussed by Indian mathematicians from the times of Āryabhaṭa. Interestingly, the trial method based on factorization has not been discussed by Bhāskarācārya. In modern times, it is well understood that factorization of large numbers is a difficult problem. Hence, the method of Bhāskarācārya has an edge over other methods which require factorization. Bhāskarācārya extends this method by way of a hint in one verse. He says, to find square root of a surd expression with 3 surds, you should collect 2 surds, for 6 surds, collect 3 surds, for 10 surds collect 4 surds and for 15 surds collect 5 surds. After this, one should follow a method in analogy with 3-surd case. Although this is the right starting point, there are sometimes difficulties in proceeding with the problem especially when there are 3 or more surds in the expression. The aim of this paper is to analyse especially the 3-surd case.

Keywords: Bhāskarācārya, quadratic surds, squares, square roots.

Introduction

IN "Karaṇīṣaḍvidha" part of his book *Bījagaṇita* (Abhyankar 1980), Bhāskarācārya tells us about six basic operations about quadratic surds, viz. addition, subtraction, multiplication, division, squaring and finding square root. For example, rationalizing the denominator of a quadratic surd is explained. Here we concentrate on the method of Bhāskarācārya about how to find a square root of an expression in quadratic surds. In high school we learn trial method of extracting square root. For example, to find square root of $29 + 2\sqrt{210}$ we find 2 factors of 210 whose sum is 29. Here $210 = 15 \times 14$ and $29 = 15 + 14$. Thus, $\sqrt{15} + \sqrt{14}$ is a square root. For large numbers, factorization can be difficult, and also we may have to try many possibilities. On the contrary, Bhāskarācārya gives a root-finding algorithm based on the methods that he has already introduced, e.g. finding square root of an integer.

वर्गे करण्या यदि वा करणयोः तुल्यानि रूपाणि अथवा बहूनाम् ।
विशोधयेत् रूपकृतेः पदेन शेषस्य रूपाणि युतोनितानि ।।

पृथक् तद् अर्धे करणीद्वयं स्यात् मूले अथ बह्वी करणी तयोः या ।
रूपाणि तानि एवम् अतः अपि भूयः शेषाः करण्यो यदि सन्ति वर्गे ।।

To find the square root of a quadratic surd expression which has one or more surds (with positive sign), from the square of the integer term subtract one or two or more integers under the radical sign. The integer that you get should be a perfect square. Take its square root. Add it to and subtract it from the integer term in the expression. Divide these two results by 2. Take the sum of the square roots of the resulting two numbers. If no surds are remaining in the original square, this is the answer. Otherwise, treat the larger of these two surds as an integer and proceed as above.

Thus, to find a square root of $5 + 2\sqrt{6}$, first write it as $5 + \sqrt{24}$. Now $5^2 - 24 = 1$ is a perfect square. Get 2 numbers $(5 + 1)/2$ and $(5 - 1)/2$, i.e. 3 and 2.

Thus, the square root is $\sqrt{3} + \sqrt{2}$.

Similarly, for $29 + 2\sqrt{210}$, first write it as $29 + \sqrt{840}$. Now, $29^2 = 841$. We have, $841 - 840 = 1$ which is a perfect square with square root 1. Then $(29 + 1)/2$ and $(29 - 1)/2$, i.e. 15 and 14 give the answer as $\sqrt{15} + \sqrt{14}$.

Note again that the method of Bhāskarācārya is quite general and it works for a general field situation, i.e. quadratic extensions of fields whose elements can be considered as quadratic for finding square root of $a + \sqrt{b}$, the method of Bhāskarācārya works. What we require is that $a^2 - b$ is a perfect square in the field. As an illustration let us find the square root(s) of a complex number $z = A + iB$.

First write $A + iB$ as $A + \sqrt{(-B^2)}$. As $-B^2$ is not a perfect square, this is a surd expression over the field of real numbers. Now, $A^2 - (-B^2) = A^2 + B^2 = |z|^2$ is a perfect square over real numbers as it is non-negative. Its square root is $\sqrt{(A^2 + B^2)} = |z|$.

Now get $[A + \sqrt{(A^2 + B^2)}]/2$ and $[A - \sqrt{(A^2 + B^2)}]/2$. The sum of the square roots of these expressions is the answer, i.e. $\sqrt{((A + |z|)/2)} + \sqrt{((A - |z|)/2)}$. This is a complex number with first term purely real and the second term purely imaginary. Here, the $\sqrt{}$ sign denotes a non-negative real root or a purely imaginary root with non-negative real coefficient of i. The other square root is the negative of this one.

It may be noted that the method of Bhāskarācārya works for finding square root of $a + \sqrt{b}$ for the general case of field extensions even if $a^2 - b$ is not a perfect square in the given field but lies in a proper extension of the field. If we denote a root of $a^2 - b$ by c, then the answer is $\sqrt{((a + c)/2)} + \sqrt{((a - c)/2)}$, and square of this quantity is $a + \sqrt{b}$. However, this answer is neither convenient, nor it is simpler than just writing $\sqrt{(a + \sqrt{b})}$. Thus, Bhāskarācārya requires that $a^2 - b$ is a perfect square and examples in which such conditions are not satisfied are called improper (*asat*).

Justification for the method of Bhāskarācārya in the case of 1 surd is as follows:

Suppose $a + \sqrt{b}$ is the square of $\sqrt{A} + \sqrt{B}$. Hence, $a + \sqrt{b} = A + B + \sqrt{4AB}$. $A + B = a$, $b = 4AB$, so $a^2 - b = (A - B)^2$. This is a perfect square.

Take its square root. $A - B$ or $B - A$. Add any one to a and also subtract it from a. That gives $2A$ and $2B$. Dividing by 2 we get A and B.

We shall in what follows discuss the method of Bhāskarācārya in the 3-surd case with justification why and to what extent the method works and point out precautions for implementation by considering various examples.

Dealing with Surd Expressions with 3 Surds

Bhāskarācārya indicates the method to be followed if the surd expression contains 3 or more surds. Although he considers surd expressions with integer numbers, his method equally applies to surds with rational numbers or even elements from a field.

Here, we first observe that if we have a surd expression with 2 terms, it is either of the type $a + \sqrt{b}$ or $\sqrt{a} + \sqrt{b}$. In both the cases the square is of the form $a + \sqrt{b}$.

A proper 3 term surd expression is of the type $a + \sqrt{b} + \sqrt{c}$ or $\sqrt{a} + \sqrt{b} + \sqrt{c}$. Here, it is assumed that no surd is an integer or rational multiple of any other, i.e. no further simplification of the surd expression is possible. The square of any such expression is of the type $a + \sqrt{b} + \sqrt{c} + \sqrt{d}$ and thus has 3 surds in it. Similarly, square of a surd expression with 4 terms has 7 terms with one integer term and 6 surds, square of a surd expression with 5 terms has one integer and 10 surds, 6 terms correspond to 15 surds and more generally n terms correspond to $C(n, 2) = n(n1)/2$ surds. For example, a square of a surd expression cannot have just 2, 3, 4, 5, 7, 8, 9, 11, etc. surds in it. It is thus clear why Bhāskarācārya proposes his method for finding square roots of surd expressions with 1, 3, 6, 10, 15 surds in the following verses:

एकादि-संकलितमितकरणीखण्डानि वर्गराशौ स्युः।
वर्गे करणीत्रितये करणीद्वितयस्य तुल्यरूपाणि ॥

करणीषट्के तिसृणां दशसु चतसृणां तिथिषु च पञ्चानाम् ।
रूपकृतेः प्रोज्झ्य पदं ग्राह्यं चेत् अन्यथा न सत् क्व अपि ॥

उत्पत्स्यमानया एवं मूलकरण्या अल्पया चतुर्गुणया ।
यासाम् अपवर्तः स्यात् रूपकृतेः तोः विशोध्याः स्युः ॥

अपवर्ते या लब्धा मूलकरण्यो भवन्ति ताः च अपि ।
शेषविधिना न यदि ताः भवन्ति मूलं तदा तद् असत् ॥

In a square there are one or more surds together. If the expression has 3 surds, we have to subtract from the square of the integer number a number equal to the sum of 2 numbers under radical sign. If it has 6 surds, 3 such should be removed. If it has 10 surds then remove 4 such. If it has 15 surds, then remove 5 such. If after removal, the difference is not a perfect square, then the example is not proper or it is *asat*.

Now consider a square expression with 3 surds. This will have an integer term too and it will be the square of an expression of the type $\sqrt{A} + \sqrt{B} + \sqrt{C}$. Here none of A, B, C will be a multiple of any other by a square of a rational number. Otherwise, the 3 terms will merge into 2 or 1 term. Then A, B, C will be non-squares or at most one of them will be a perfect square.

Example: Recall the example $5 + \sqrt{24}$. $25 - 24 = 1$ is a square. Square root is 1.

$(5 + 1)/2$ and $(5 - 1)/2$ gives 3 and 2. So the square root of $5 + \sqrt{24}$ is $\sqrt{3} + \sqrt{2}$.

Now for an illustration of B method in the above verses, consider a square with 3 surds.

Example: $(\sqrt{3} + \sqrt{5} + \sqrt{7})$ 2 = $15 + \sqrt{60} + \sqrt{140} + \sqrt{84}$.

To find square root of $15 + \sqrt{60} + \sqrt{140} + \sqrt{84}$

Step I: Take any two surds together, e.g. $\sqrt{60}$ and $\sqrt{140}$. Take $a = 15$, $b = 60 + 140 = 200$.

Imitate the procedure of finding the square root of $a + \sqrt{b}$.

$(15)^2 = 225$. $225 - 200 = 25$. This is a perfect square. The square root is 5. By the method explained, $(15 + 5)/2 =10$, $(15 - 5)/2 = 5$. Thus, the square root of $15 + \sqrt{200}$ is $\sqrt{10} + \sqrt{5}$.

Step II: Out of this reserve, the smaller one, viz. 5 as comprising a part of the final answer as $\sqrt{5}$.

Go ahead with 10 as an integer and consider the remaining surd, here 84 along with it, i.e. consider $10 + \sqrt{84}$ as before.

$100 - 84 = 16$. Square root = 4. $(10 + 4)/2 = 7$, $(10 - 4)/2 = 3$. Thus, the square root of $10 + \sqrt{84}$ is $\sqrt{7} + \sqrt{3}$. Using the reserved $\sqrt{5}$, the method of B tells us that the final answer is $\sqrt{5} + \sqrt{7} + \sqrt{3}$ as expected.

It may be noted in this problem that if we take any two of the 3 surds, then also the method works.

Take, for instance 60 and 84 from $15 + \sqrt{60} + \sqrt{140} + \sqrt{84}$. Now, by the above method, $60 + 84 = 144$, $225 - 144 = 81$. Square root is 9. $(15 + 9)/2 = 12$, $(15 - 9)/2 = 3$. Reserve 3. Go ahead with 12. Consider $12 + \sqrt{140}$. $144 - 140 = 4$. Square root is 2. $(12 + 2)/2 = 7$, $(12 - 2)/2 = 5$. Answer $\sqrt{7} + \sqrt{5} + \sqrt{3}$.

Try 140 and 84. $140 + 84 = 224$. $225 - 224 = 1$. Square root = 1. $(15 + 1)/2 = 8$, $(15 - 1)/2 = 7$.

Reserve 7. Go ahead with 8. Consider $8 + \sqrt{60}$. $64 - 60 = 4$. Square root = 2. $(8 + 2)/2 = 5$, $(8 - 2)/2 = 3$. $\sqrt{5} + \sqrt{3} + \sqrt{7}$ is the answer.

The reason why any two surds can be taken together in this method can be explained as follows: Suppose $a + \sqrt{b} + \sqrt{c} + \sqrt{d} = (\sqrt{A} + \sqrt{B} + \sqrt{C})^2$.

Then $a + \sqrt{b} + \sqrt{c} + \sqrt{d} = A + B + i + \sqrt{(4AB)} + \sqrt{(4AC)} + \sqrt{(4BC)}$. Then $a = A + B + C$.

Take, for instance, b = $4AB$, $c = 4AC$. Then $d = 4BC$.

As per the method, in Step I, taking together first two surds, consider $a^2 - b - c$.

This becomes $(A + B + C)^2 - 4AB - 4AC = (-A + B + C)^2$, which is a perfect square.

The square roots are $-A + B + C$ and $A - B - C$. Taking any of these roots, by the operation $(a + \text{root})/2$ and $(a - \text{root})/2$, we get $B + C$ and A. Now in Step II, take the larger one of these two (*bahvī karaṇī*) as per the initial verses of Bhāskarācārya above, which is expected to be $B + C$.

And here is the catch. Here for proceeding algebraically, we must keep A for reserve for the final answer and go ahead with $B + C$. It is likely that actually the smaller number is $B + C$. So to get the correct answer we have to proceed with $B + C$. So larger of the two may not always work, contrary to the explanation of Bhāskarācārya. Thus, "one of them works" is the correct way of putting. Go ahead with $B + C$. Then consider $B + C$ and the remaining surd $\sqrt{4BC}$. Thus, we get $B + C + \sqrt{4BC}$. By the method, $(B + C)^2 - 4BC = (B - C)^2$. The square roots are $B - C$ and $C - B$. To $B + C$ add any root and also subtract the root from $B + C$. Divide by two. This gives B and C. Using A in reserve, the answer is $\sqrt{A} + \sqrt{B} + \sqrt{C}$.

Since AB and AC have A common, AB and BC have B common, and BC and AC have C common, the method will work with the choice of any two surds of the three, provided at the end of Step I, we make the right choice of the number for working in Step II.

To illustrate the problem in the last step we consider

$(\sqrt{10} + \sqrt{2} + \sqrt{3})^2 = 15 + \sqrt{80} + \sqrt{120} + \sqrt{24}$.

Now $225 - 80 - 120 = 25$. Square root 5. $(15 + 5)/2 = 10$, $(15 - 5)/2 = 5$. Here, if we keep 5 reserve as it is smaller and proceed with 10, we have to consider $10 + \sqrt{24}$.

Here we get $100 - 24 = 76$ which is not a perfect square, so the method fails. On the contrary, reserving the larger number 10 for the final answer and proceeding with 5, we have to consider $5 + \sqrt{24}$. Then $25 - 24 = 1$ which is a perfect square with square root 1. $(5 + 1)/2$ and $(51)/2$ give 3 and 2. So we get $\sqrt{10} + \sqrt{3} + \sqrt{2}$ as the correct answer.

Thus, in this problem, one of the two choices obtained in the first step works, but not the other. In such an example we can get the final answer by selecting the choice at the end of Step I which takes us to the final answer and reject the other choice.

A Misleading Example

Here, we consider an example which gives an answer for both the choices obtained in Step I without internal contradiction. One answer is correct and the other is not. Take for instance $(\sqrt{35} + \sqrt{6} +$

$\sqrt{11})^2 = 52 + \sqrt{840} + \sqrt{1540} + \sqrt{264}$. Taking first 2 surds together, 2705 – 840 – 1540 = 324. Square root is 18. The two numbers obtained are (52 + 18)/2 = 35, (52 – 18)/2 = 17. Keep 17. In Step II, go ahead with 35 (larger number). We get $35 + \sqrt{264}$. 1225 – 264 = 961. Square root is 31. We get 2 numbers (35 + 31)/2 and (35 – 31)/2, i.e. 33 and 2. Everything works without any contradiction. All required numbers are perfect squares and we think we have arrived at the answer $\sqrt{33} + \sqrt{2} + \sqrt{35}$. Unfortunately, this is a wrong answer, as can be checked by squaring the expression. But that does not mean that the given expression is not a perfect square. On the contrary, going ahead with 17, we get, $289 - 264 = 25 = 5^2$, and (17 + 5)/2 = 11 and (17 – 5)/2 = 6, giving us $\sqrt{35} + \sqrt{6} + \sqrt{11}$ as the right answer. This example illustrates that even if the internal required numbers are perfect squares, it may lead to a wrong answer.

In an example like $10 + \sqrt{40} + \sqrt{60} + \sqrt{24}$, taking first two surds we get 100 – 40 – 60 = 0, so (10 + 0)/2 and (10 – 0)/2 give the same values 5, 5. So there is no difficulty here. Keeping one 5 as reserve and going ahead with 5 we get $\sqrt{3} + \sqrt{2} + \sqrt{5}$ as the answer.

In conclusion, the method of Bhāskarācārya works with a precaution. The statement of Bhāskarācārya about taking larger number in Step II does not always work:

> पृथक् तद् अर्धे "करणीद्वयं स्यात् मूले अथ बह्वी करणी तयोः या ।
> रूपाणि तानि एवम् अतः अपि भूयः शेषाः करण्यो यदि सन्ति वर्गे" ।।
>
> Take the sum of the square roots of the resulting two numbers. If no surds are remaining in the original square, this is the answer. Otherwise, treat the larger of these two surds as an integer and proceed as above.

So at the end of Step I we get 2 numbers and one of them certainly works for going ahead when a perfect square surd expression with 3 surds is given. The number does not work may be clearly in the intermediate steps when we do not get a required number as a perfect square. Sometimes all the required intermediate numbers are perfect squares, still the answer is wrong. Hence after getting the answer tallying is necessary and if required we should use the other number in Step I for proceeding in Step II.

Examples Given by Bhāskarācārya in 3-Surd Case

In "Karaṇīṣaḍvidha" part of *Bījagaṇita*, Bhāskarācārya asks four problems in the 3-surd case. His first problem on 3-surd expression is:

वर्गे यत्र करण्यः दन्तैः सिद्धैः गजैः मिताः विद्वन् ।
रूपैः दशभिः उपेताः किं मूलं ब्रूहि तस्य स्यात् ।।

Oh learned! find the square root of $10 + \sqrt{32} + \sqrt{24} + \sqrt{8}$.

Here $100 - 32 - 24 = 44$ is not a perfect square, so the expression is not a perfect square. Note that as explained earlier, it is not necessary to try other pairs of surds. Even if we do, $100 - 24 - 8 = 68$ and $100 - 32 - 8 = 60$ are not perfect squares.

The next example is:

वर्गे यत्र करण्यः तिथिविश्वहुताशनैः चतुर्गुणितैः ।
तुल्या दशरूपाढ्याः किं मूलं ब्रूहि तस्य स्यात् ।।

What is the square root of $10 + \sqrt{60} + \sqrt{52} + \sqrt{12}$.

Here $100 - 60 - 52$ is negative, so not a square. So the expression is not a perfect square, although $100 - 52 - 12 = 36$ is a perfect square. Actually proceeding with 36 one gets $(10 + 6)/2$ and $(10 - 6)/2$, i.e. 8 and 2. We cannot proceed with 2 as $4 - 60$ is negative. Proceeding with 8, we get $8 + \sqrt{60}$. Then $64 - 60 = 4 = 2^2$. $(8 + 2)/2 = 5$ and $(8 - 2)/2 = 3$. $\sqrt{5} + \sqrt{3} + \sqrt{2}$ is the expected square root, but it is not as can be directly checked. Since for a perfect square surd expression, differences obtained from any two surds should be perfect squares, it is enough to get one difference not a perfect square.

Bhāskarācārya is testing the reader with another problem in the following verse:

अष्टौ षट्पञ्चाशत् षष्टिः करणीत्रयं कृतौ यत्र।
रूपैः दशभिः उपेतं किं मूलं ब्रूहि तस्य स्यात् ।।

Find the square root of $10 + \sqrt{8} + \sqrt{56} + \sqrt{60}$.

Consider the surd expression $E = 10 + \sqrt{8} + \sqrt{56} + \sqrt{60}$. Here we take first 2 surds together. Consider $100 - 8 - 56 = 36$ which is a square. Consider (10 + 6)/2 and (10 – 6)/2, i.e. 8 and 2. If we keep 8 as reserve and take 2 for further analysis, we get $2 + \sqrt{60}$. But 4 – 60 is negative, so we have to abandon this. Take the larger integer 8 for further analysis and reserve 2 as a part of the answer in the form of $\sqrt{2}$. Unused surd is $\sqrt{60}$. We are left with $8 + \sqrt{60}$. This is going to be a perfect square of a surd expression. We have $8^2 = 64$. 64 – 60 = 4, which is a perfect square, with square root 2. Then (8 + 2)/2 and (8 – 2)/2 give 5 and 3. By Bhāskarācārya's method this gives $\sqrt{5} + \sqrt{3}$ as part of the answer. Final answer is obtained using previous $\sqrt{2}$. Thus, the square root is $\sqrt{5} + \sqrt{3} + \sqrt{2}$. However, after tallying we see that the square of $\sqrt{5} + \sqrt{3} + \sqrt{2}$ is not the original expression. This happens because the given expression *E* is not a square of such a surd expression. This will be more clear when we take the 2nd and 3rd surd in *E*. If we work out 100 – 56 – 60, we get – 16 which is a negative number and a non-square. Also, 100 – 8 – 60 = 32 is a non-square. This illustrates that the method can mislead the reader if you start with a surd expression which is not a perfect surd square.

After giving enough warning to the readers by these three verses that blindly following the method is not useful and the expression may not be a perfect square surd expression, Bhāskarācārya gives one problem in which the expression is a perfect square.

चत्वारिंशद्-अशीति-द्विशती-तुल्याः करण्यः चेत् ।
सप्तदशरूपयुक्ताः तत्र कृतौ किं पद ब्रूहि ।।

Find the square root of $17 + \sqrt{40} + \sqrt{80} + \sqrt{200}$.

Here, 289 – 40 – 80 = 169 is a square with square root 13. (17 + 13)/2 = 15 and (17 – 13)/2 = 2. Keeping 2 as reserve and proceeding with 15, we get $15 + \sqrt{200}$. Now, 225 – 100 = 25 = 52. (15 + 5)/2 = 10, (15 – 5)/2 = 5. Thus, the expected square root is $\sqrt{10} + \sqrt{5} + \sqrt{2}$. The reader is now careful and checks that the square of this expression is indeed the given expression.

Remark

Although in the 3-surd case, all the three differences, obtained from a square surd expression by taking any 2 surds out of the 3, are perfect squares, this is no longer the case when there are 6 surds in the expression. In that case and in later cases with 10 surds, 15 surds, etc. mentioned by Bhāskarācārya, even if the expression is a perfect square, in the method of Bhāskarācārya only certain 3 surds, 4 surds, 5 surds, etc. have to be taken together for subtraction from the square of the integer.

Reference

Abhyankar, S.K., 1980, *Bhāskarācārya's Bījagaṇita and Its Translation*, Pune: Bhaskaracharya Pratishthana.

14

The Fore-Shadowing of Banach's Fixed-Point Theorem

Among Indian and Islamic Mathematicians: Procedural or Spatial Intuition?

Johannes Thomann

IN MATHEMATICS, the way from conjecture to proof can be long. Fermat's Last Theorem is famous. It was published in 1637 CE, not as a conjecture, but as a lemma for which Fermat claimed to have found a wonderful proof (*demonstrationem mirabilem sane detexi*). Unfortunately, the place for notes in the margin was not large enough to write it down (*Hanc marginis exiguitas non caperet*). The decisive proof of the theorem was published in 1995 by Andrew Wiles, 358 years after Fermat's claim. Today, nobody believes that Fermat's alleged proof was a valid one, but since he was such an eminent expert in number theory, he might have had a presentiment of something which seemed to make the theorem evident. In the following, a case will be described in which the time interval between the intuition of a lemma and its proof is even longer, in fact, more than a millennium.

Banach's Fixed-Point Theorem

Metric spaces are a core topic in modern mathematics. One of the most frequently used lemmata is the fixed-point theorem, named after the famous Polish mathematician Stefan Banach (1892–1945 CE):

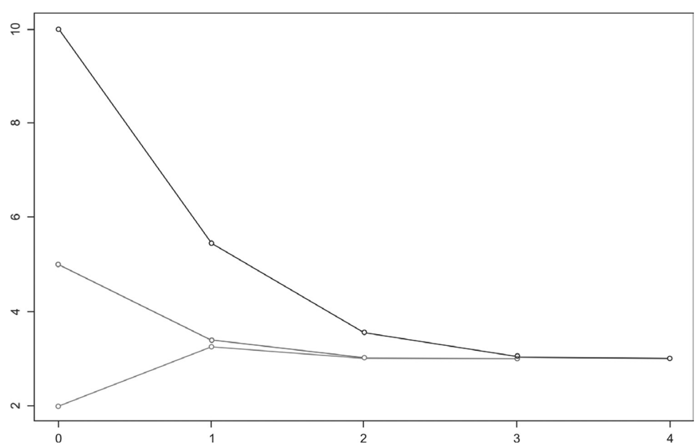
> A contraction mapping T of a complete metric space on itself has a unique fixed-point x^*, which can be constructed by the iteration $x_n = T(x_{n-1})$, starting with an arbitrary element x_0.

A simple example may serve as a demonstration. The real numbers form a complete metric space, and the so-called Heron method for extracting square roots demonstrates the construction of the fixed-point by iteration:

$$x_{n+1} = x_n - (x_n^2 - a)/(2x_n).$$

This converges towards the square root of a. If $a = 9$ and the start value $x_0 = 10$, $x_0 = 5$ and $x_0 = 2$, one obtains the following values for x_1, x_2, x_3 and x_4:

Start Value x_0	*1st Iteration x_1*	*2nd Iteration x_2*	*3rd Iteration x_3*	*4th Iteration x_4*
10	5.45	3.550688	3.042704	3.00030000000
5	3.40	3.023529	3.000092	3.00000000100
2	3.25	3.009615	3.000015	3.000000000039



This method was already used by the Babylonian mathematicians. It converges with any positive value of a and any start value x_0.

In classical antiquity, iterations were occasionally used for other and more complex problems. Ptolemy used in the Almagest an iterative method for the calculation of the true conjunction of the sun and the moon. It seems that for him, iterative methods were only the last choice when everything else did not work.

In India, the attitude of mathematicians towards iterative methods was different. They used them for all kind of problems. Iterative methods used by Indian mathematicians fall into two categories: "fixed-point" and "two-point" techniques (Plofker 2002: 168). Both were called indistinctly *asakṛt* (not just once). The best known "two-point" technique is the *Regula falsi*. In the following, the focus will be on the "fixed-point" techniques.

Like in the Mediterranean and the Middle East, one of the most simple iterative methods in Indian mathematics is the extraction of square roots. A formula, somewhat different from that of Heron, was used at least from about 500 CE onward. But in astronomy, iterative methods were used even in cases where analytical solutions were available (ibid.: 170).

Solar Eclipse Calculation according to Brahmagupta

From the many cases in which Brahmagupta (598 – after 665 CE) used iterative methods we take that of solar eclipse calculation. In chapter 4 of his *Khaṇḍakhādyaka* he describes his method how to find the true conjunction of the sun and the moon by an iterative technique:

> Multiply the *jyā* (sine) of the moon's *mandakendra* (mean anomaly) by the *jiā* of its *natakāla* (hour angle). Multiply the product again by 499 and divide by the square of the *trijyā* (radius). The result is in seconds. If the *mandaphala* (equation of the centre) of the moon is subtractive, add or subtract the result to or from its corrected longitude, according as it is in the eastern or western half of the sky. If the *mandaphala* is additive, subtract the result from the corrected longitude of the moon, whether it is in the eastern or western half of the sky (the result is its correct true longitude). The process should be repeated till the longitudes are fixed.
>
> – Tr. Chatterjee 1970: 80

In the present context, it is not necessary to go into the details of this calculation. What interests us here is the last instruction. Brahmagupta did not specify how many iterations should be made. He rather formulated a criterion "till the longitudes are fixed", and that shows that he had a clear concept of convergence. The method works well, because the equation of the centre is small compared to the motion in mean longitude.

Solar Eclipse Calculation according to Ḥabash al-Ḥāsib

Ḥabash al-Ḥāsib (d. after 869 CE), whose origin was in the Central Asian town Marw, lived in Baghdad, Damascus and Samarrā. Two Zījes, attributed to him, exist (Debarnot 1987; Thomann 2010). The earlier of the two is called *al-Zīj al-dimashqī* (the Damascus tables). In the calculation of a solar eclipse he describes an iterative method in detail:

> Thus, after that, we enter the reminder into the column of the [argument] numbers [of the table]. What we find [in the column] next to it in degrees and minutes is the lunar parallax, and it is the *first* parallax. We add it to the true distance of the sun from the true mean heaven.[1] We enter with the result of the true distance of the sun to which we have added [the parallax] into the column of the [argument] number. We take what we find [in the column] next to it in degrees and minutes, the *second* parallax. We add it to the true distance of the sun. We enter the result of the true distance of the sun with the addition of the second parallax into the column of the [argument] number. We take what we find [in the column] next to it in degrees and minutes, the third parallax. We add it to the true distance of the sun. Next we enter the true distance of the sun with what we have added – degrees and minutes of the *third* parallax – into the column of the [argument] number. We take what we find [in the column] next to it in degrees and minutes, the *fourth* parallax. We add it to the true distance of the sun. Next we enter the true distance of the sun with the addition of the fourth parallax into

[1] "Mean heaven" is not the intersection of the ecliptic with the meridian, but the point of the ecliptic with the maximum altitude; cf. Kennedy 1956: 49.

> the column of the [argument] value. We take what we find [in the column] next to it in degrees and minutes, the *fifth* parallax, and we call it "degrees of the smallest distance".[2]

In this case, the approximation serves the calculation of the place of conjunction corrected for the lunar parallax. The method works well, because the parallax is small compared to the motion in longitude. Ḥabash insists on five iterations, which is far more than necessary. In other cases, he recommends only three iterations. The five iterations might have been motivated by the fact that in the case of solar eclipses precision is crucial.

Ḥabash's method for calculating eclipses is entirely different from that found in the Almagest (Kennedy 1956: 51). Ḥabash knew and admired Ptolemy's Almagest. However, in many points he followed the methods of Indian astronomers, which were known through translations into Arabic. He did use Indian trigonometric functions sine and cosine throughout, and never used the Greek methods with chords (Thomann 2013: 546). For calculations with great numbers, he used Hindu-Arabic numerals (ibid.: 545-46). In the preface to the work, he mentions two Indian works by name, *al-Sindhi[n]d* and *al-Arkand*, both being adaptations of works by Brahmagupta (ibid.: 547-48). He seems to have had a special interest in Indian mathematics and astronomy, and he must have had some access to original Sanskrit material. In his chapter on the lunar mansions he provided a table with the Sanskrit names of the twenty-seven *nakṣatra*s transliterated in Arabic script, together with their Arabic equivalents (ibid.: 548-52). Therefore, it is likely that he followed also Indian methods in his iterative technique for calculating solar eclipses. If one compares his description to that of Brahmagupta, some differences are conspicuous. Brahmagupta's description is very brief, and the instruction for the iteration is laconic. In contrast, Ḥabash's instructions are verbose and avoids any abbreviations. Furthermore, he follows the Greek style of addressing his readers by "we" in the first person plural, while Brahmagupta addresses them by "you" in the second person

[2] Translation by the author. See the Appendix with the Arabic text, transcribed from MS Istanbul, Süleymaniye Kütüphanesi, Yeni Cami 784, ff. 210v-211r.

singular. The two styles correspond to the different traditions of teaching astronomy (ibid.: 510-13). In ancient Alexandria, scholars delivered lectures before a larger audience in lecture halls. The Indian tradition was that a scholar thought a pupil, who lived in his house, face to face. But common to both texts is the procedural approach, in which the technique is described only step by step without theoretical explanations, leave alone proofing arguments.

One may assume with some confidence that Ḥabash was indeed inspired by Indian sources in his iterative technique, eventually by a work of Brahmagupta.[3]

The Kind of Intuition at Work for Creating Iterative Methods

The question which remains is: what kind of intuition was it which lead to such solutions, as described above? If one goes back to Banach's fixed-point theorem, in most accounts of it, the term "contracting mapping" is used, and the structure on which the mapping is executed is called "space". This points clearly to spacial intuition by which the theorem can be understood. But it is another question if eighth- and ninth-century astronomers were thinking alike. In favour of spacial intuition in astronomical reasoning in that time a slightly later author can give evidence. The tenth-century astronomer al-Qabīṣī (d.967 CE) wrote a treatise on the examination of astronomers and astrologers. At the beginning he goes on to describe the different level of competence in astronomy. The highest level is obtained by the perfect astronomer who knows all the proofs of the Almagest, and who is able to establish astronomical tables based on his own observations. Most relevant in our context is the second level. The astronomer who has reached it, is able to form a mental image of the heaven at any time, but is not able to prove it (Thomann 2017: 926). Such an ability seems indeed a possible base for inventing iterative methods. If the inventor was able to make the step from mean longitude to true longitude, or from true position to apparent position corrected for parallax, in a mental image, the idea of iterative approximation would be well in range.

[3] This has already been assumed by Kennedy/Transue 1956: 83.

However, there are reasons which speak against that explanation. In contrast to Greek astronomical works, in which geometrical arguments are omnipresent, Indian works of the earlier epoch lack such an approach. The techniques are explained by steps of calculation. Numerical values are transformed by arithmetical and trigonometric functions. Ḥabash followed this approach throughout, and, as has been said, did not provide geometrical proves. A caveat must be made. The manuscript of early Sanskrit works on astronomy do not contain geometrical drawings. However, Brahmagutpa refers in the *Khaṇḍakhādyaka* to a drawing to be made in order to represent the situation of an eclipse (Chatterjee 1970: 81-85; Plofker 2002: 98-102). In a later manuscript, a rudimentary eclipse diagram is extant (Plofker 2002: 102, *fig*. 4.12). Only a few drawings are found in the Zīj of Ḥabash, but at least their usefulness for understanding complex situation is acknowledged.[4]

Perhaps it is wrong to present procedural and spacial intuition as an alternative. A possible strategy could have been to combine both forms of intuition, the use of a mental image, eventually sustained by drawings, and the observation of a series of numerical results obtained by calculation. Brahmagupta's criterion "till the longitudes are fixed" points to an experience in calculation with a fixed number of fractional positions.

In the case of extracting square roots one could think of algebraic reasoning. In the first estimate, x the unknown error may be e; then

$$S = (x + e) \wedge 2$$

$$S = x \wedge 2 + 2xe + e \wedge 2$$

$$e = (S - x\wedge2)/(2x + e)$$

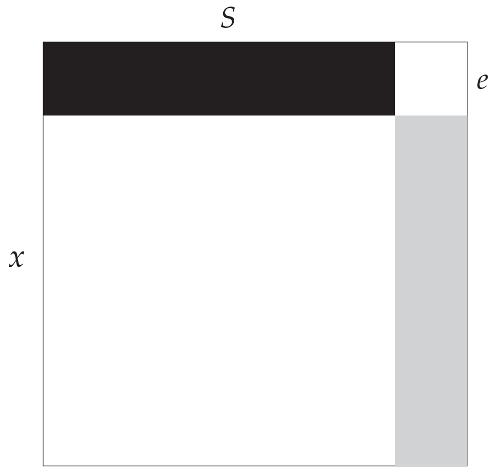
then the error can be estimated by

$$e \approx (S - x \wedge 2)/(2x)$$

since e is small compared to x. The new estimate of x is:

$$x \text{ revised} = x + (S - x \wedge 2)/2x.$$

[4] MS Istanbul, Süleymaniye Kütüphanesi, Yeni Cami 784, ff. 163r, 165r.



The same reasoning can also be geometrically obtained:

If the square with the side-length e is neglected, the red rectangles divided by the longer sides $2x$ are an estimate for e.

However, while an intuitive geometrical reasoning can lead to the iterative technique in this simple case, in other more complicated cases this would not work anymore. The same holds for the algebraic approach. A function like $\phi(x) = b + k \sin x$, which was iteratively solved by Ḥabash, is a transcendental function, and was the object of many studies from the seventeenth to the twentieth centuries CE (Dutka 1997).

The consideration made so far are neither unambiguous, nor final. More examples should be examined, and spacial and procedural concepts should be drafted, which could have lead to the invention of the technique in question. The aim of this paper was only to point to the problem and to initiate a discussion on it in the hope to give clearer answers in the future.

Despite the rather negative result so far, amore general conclusion can be made. If one is looking for a real fore-shadowing of Banach's fixed-point theorem as a general method for developing iterative solutions, one has to look at Indian works on mathematics and astronomy. The variety of problems which were solved by iterative techniques using the fixed-point approach shows that recursive functions were the offspring of a general notion of contraction mapping. This was another great achievement which Indian mathematicians brought forward, and which was spread to the West by mathematicians of the Islamic world.

Appendix: Arabic Text of Ḥabash's Description of the Iterative Calculation of the Apparent Place of the Sun

The text is transcribed from the manuscript Istanbul, Süleymaniye Kütüphanesi, Yeni Cami 784, ff. 210v-211r.

قئاقدو جرد نم هتلابق دجن ام ددعلا رطس يف كلذ دعب يقابلاب لخدن مّث

دعب ىلع هديزتف لوّألا رظنملا فالتخا كلذو رمقلا ؟رظنم/رطيم فالبخا
يقّحلا سمشلا

اندز امعم يقّحلا سمشلا دعب غلبمب لخدنو يقّحلا ءامسلا طسو نم211r
[ذـ]ـخأنو ددعلا رطس يف هيلع

دعب ىلع هديزنف ؟يناثلا رظنملا فالتخا قئاقدو جرد نم هتلابق دجن ام
[..] يقّحلا سمشلا

يف يناثلا رظنملا فالتخا ةدايز عم يقّحلا سمشلا دعب غلبمب لخدن
[هتل]ابق دجن ام ذخأنف ددعلا رطس

يقّحلا سمشلا دعب ىلع هديزنو ثلاثلا رظنملا فرلتخا قئاقدو جرد نم
فالتخا ةدايز عم

جرد نم هيلع اندز امعم يقّحلا سمشلا دعب غلبمب لخدن مّث يناثلا رظنملا
رظنملا فالتخا قئاقدو

فالبخا قئاقدو جرد نم هتلابق دجن ام ذخأنو ددعلا رطس يف ثلاثلا
عبارلا رظنملا

عم يقّحلا سمشلا دعب غلبمب لخدن مّث يقّحلا سمشلا دعب ىلع هديزنو
رظنملا فالتخا ةدايز

فالتخا قئاقدو جرد نم هتلابق دجن ام ذخأنو ددعلا رطس يف عبارلا
رظنملا

رغصألا دعبلا جرد هيمسنو سماخلا

References

Chatterjee, Bina (ed.), 1970, *The Khaṇḍakhādyaka* (*An Astronomical Treatise*) *of Brahmagupta*; *with the Commentary of Bhaṭṭotpala*, Calcutta: World Press.

Debarnot, M.-Th., 1987, "The Zīj of Ḥabash al-Ḥāsib: A Survey of MS Istanbul Yeni Cami 784/2", *Annals of the New York Academy of Sciences*, 500: 35-69.

Dutka, Jacques, 1997, "A Note on 'Keplers's Equation'", *Archive for History of Exact Sciences*, 51(1): 59-65.

Kennedy, E.S., 1956, "Parallax Theory in Islamic Astronomy", *Isis*, 47 (1): 33-53.

———, 1969, "An Early Method of Successive Approximations", *Centaurus*, 13(3-4): 248-50.

Kennedy, E.S. and W.R. Transue, 1956, "A Medieval Iterative Algorism", *The American Mathematical Monthly*, 63(2): 80-83.

Plofker, Kim, 2002, "Use and Transmission of Iterative Approximations in India and the Islamic World", in *From China to Paris: 2000 Year Transmission of Mathematical Ideas*, ed. Y. Dold-Samplonius et al., pp. 167-86, Stuttgart: Franz Steiner Verlag.

Thomann, J., 2010, "Die zwei Versionen der astronomischen Tafeln von Ḥabaš al-Ḥāsib: Ein editorisches Probelm", in *DOT Marburg* 20-24 September 2010, Zurich: online version: http://www.zora.uzh.ch/46853.

———, 2013, "Explicit and Implicit Intercultural Elements in the Zīj of Ḥabash al-Ḥāsib", in *Islam and Globalisation: Historical and Contemporary Perspectives: Proceedings of the 25th Congress of l'Union Européenne des Arabisants et Islamisants*, ed Agostino Cilardo, pp. 541-54, Leuven: Peeters Publishers.

———, 2014, "From Lyrics by al-Fazārī to Lectures by al-Fārābī: Teaching Astronomy in Baghdād (750–1000 CE)", in *The Place to Go: Contexts of Learning in Baghdād, 750-1000 CE*, ed. Jens Schiner and Damjen Jonas, pp. 503-25, Middlesex: Darwin Press.

———, 2017, "The Second Revival of Astronomy in the Tenth Century and the Establishment of Astronomy as an Element of Encyclopedic Education", *Asiatische Studien*, 71(3): 907-57.

15

Arithmetic Progression

On Comparing Its Treatment in Old Sanskrit Mathematical Texts and Modern Secondary School Curriculum in India

Medha S. Limaye

Abstract: Quite a few topics in mathematics at secondary school level have a continued existence since long time in India. That means they are rooted in the then used famous Sanskrit texts composed and refined during 500–1400 CE period. The topic of arithmetic progression currently prescribed for the standard 10 across the three major educational boards in India, viz. SSCE, CBSE and ICSE, is an example. This paper aims to evaluate the treatment given to this topic in medieval Sanskrit texts and that in the modern textbooks. The focus is to compare and contrast the method of exposing the concept, developing solution techniques and building numerical problems.

Keywords: Arithmetic progression, magic squares, numerical problems, *śreḍhī-kṣetram*.

Introduction

INDIA has a long history of teaching and learning mathematics. Quite a few topics in mathematics at secondary school level have a continued existence since long time in India. They are rooted in

Sanskrit texts composed and refined during 500–1400 CE period. The topic of arithmetic progression currently prescribed for the standard 10 across the three major educational boards in India, viz. SSCE, CBSE and ICSE, is an example. It is observed that this topic has been handled in a variety of ways in the Sanskrit texts like the *Āryabhaṭīya*, *Brāhmasphuṭasiddhānta*, *Pāṭīgaṇita*, *Gaṇitasāra-Saṁgraha*, *Līlāvatī* and *Gaṇita-Kaumudī.*

This paper evaluates the treatment given to this topic in those medieval texts and that in the modern textbooks. The focus is to compare and contrast the method of exposing the concept, developing solution techniques and building numerical problems.

Arithmetic Progression in Modern Secondary School Textbooks

Arithmetic progression is defined as a sequence of numbers such that the difference between the consecutive terms is constant.

The general form of an arithmetic progression is a, $a + d$, $a + 2d$, $a + 3d$ and so on.

The sum of a finite sequence is called an arithmetic series.

Basic formulae in this topic are derivation of the n^{th} term and sum of the first n terms.

$$t_n = a + (n - 1)d$$

and

$$S_n = n/2\ [2a + (n - 1)d].$$

Here a = first term, d = common difference, n = number of terms.

Two more formulae are also given in textbooks, viz.:

Mean term = $\frac{1}{2}(a + t_n)$ and $S_n = n[\frac{1}{2}(a + t_n)]$.

These results are derived by method of induction in modern texts. Further, a number of numerical problems are given for application of these rules in solving daily life problems.

Śreḍhīvyavahāra in Sanskrit Texts

Śreḍhī is the term for a progression in Sanskrit texts. *Śreḍhī-vyavahāra* meant determination of progression. श्रेढी resembles a staircase. Hindi word सीढ़ी and Marathi word शिडी are similar to it. Sanskrit word श्रेणी means a sequence. But the word श्रेढी became popular in practice. According to the *Buddhivilāsinī* commentary on the *Līlāvatī*, it is व्यावहारिकीयं संज्ञा। There it is said that the term is employed by the older authors for any set of distinct substances put together.

भिन्नं भिन्नं यत्किञ्चिद् द्रव्यादिकमेकीक्रियते तच्छ्रेढीत्युच्यते वृद्धैः। The mention of "older authors" suggests that the concept was known for a long period and the word द्रव्यादिकम् suggests that it was being used mainly in the context of wealth. The words सर्वधनम्, अन्त्यधनम्, आदिधनम्, मध्यधनम् used for different terms also suggest calculation of wealth. Old Sanskrit texts use the words आदि, मुख, वदन and other synonyms of face for the first term, चय, प्रचय, उत्तर for common difference, गच्छ for the number of terms, अन्त्यधनम् for the last term; मध्यधनम् for the mean term and सर्वधनम्, श्रेढीफलम्, गणितम् for the sum of all terms in a finite arithmetic progression. Sanskrit texts of medieval period dealt with both arithmetic and geometric progressions.

Terminology Used in Modern Texts

Basic term श्रेढी is retained in modern vernacular medium texts. Marathi texts use the term अंकगणिती श्रेढी for an arithmetic progression. NCERT Hindi textbooks use the term समान्तर श्रेढी for an arithmetic progression. NCERT Hindi textbooks use the terms पद for term, योग for sum and सार्व अंतर for common difference. Marathi textbooks use the words पद, बेरीज and साधारण फरक respectively for them. Mean term is not considered important in modern texts.

Development of Solution Techniques in Sanskrit Texts

Sanskrit texts state the rules in *sūtra*s with great economy in verse. The authors of original texts or the commentators explain the rules with illustrative examples. The solution techniques are *sthāpanam* (statement), *karaṇam* (solution) and sometimes *pratyayam* (verification). Formal rules and numerical problems based on arithmetic progression occur in the Bakśāli manuscript, the *Āryabhaṭīya*, *Brāhmasphuṭasiddhānta*, *Pātīgaṇita*, *Gaṇitasāra-Saṁgraha*, *Līlāvatī* and *Gaṇita Kaumudī*.

RULES IN ĀRYABHAṬĪYA

Āryabhaṭa gives the rules in two verses. The following verse gives the method to find arithmetic mean and the sum of all terms of an arithmetic progression.

> इष्टं व्येकं दलितं सपूर्वमुत्तरगुणं समुखमध्यम्।
> इष्टगुणितमिष्टधनं त्वथवाद्यन्तं पदार्धहतम्।। – v. 19

Here if $S = a + (a + d) + (a + 2d) + \ldots$ to n terms, then the steps are as follows:

i. diminish the given number of terms (n) by one,
ii. divide $(n - 1)$ by two,
iii. increase by number of the preceding terms p, i.e. $[(n - 1)/2 + p]$,
iv. multiply by the common difference (d), i.e. $[(n - 1)/2 + p] \times d$,
v. increase by the first term (a) of the whole series, i.e. $a + [(n - 1)/2 + p] \times d$,
vi. the result is the arithmetic mean,
vii. multiply this arithmetic mean by the number of terms to get the sum $= n \times \{a + [(n - 1)/2 + p] \times d\}$,
viii. if $p = 0$ then arithmetic mean $= a + [(n - 1)/2] \times d$ and $S = n\{a + [(n - 1)/2] \times d\}$, and
ix. alternatively multiply the sum of the first and the last terms (A and L) by half the number of terms, i.e. $S = n/2[A + L]$.

RULES GIVEN IN BRĀHMASPHUṬASIDDHĀNTA

The rules given by Brahmagupta and the rest of the mathematicians for finding the sum, mean and last term of an arithmetic progression are substantially equivalent to those given by Āryabhaṭa.

Brahmagupta gives the rule in the following verse:

> पदमेकहीनमुत्तरगुणितं संयुक्तमादिनाऽन्त्यधनम्।

आदियुतान्त्यधनार्धं मध्यधनं पदगुणं गणितम्।।
– *ब्राह्मस्फुटसिद्धान्त* XII.17

The period less one, multiplied by the common difference being added to the first term is the amount of the last term, i.e. $a + (n - 1) \times d = L$.

Half the sum of last and first term is the mean amount, which is multiplied by the period is the sum of the whole, i.e. $S = n/2[A + L]$.

RULES GIVEN IN GAṆITA-SĀRASAṀGRAHA

Mahāvīrācārya too gives the same basic rules. But his method of exposing the concept and developing solution techniques is different. In the first chapter Parikarmavyavahāra, he gives two distinct operations. He uses the term *saṅkalita* – addition for summing the terms of a sequence beginning with the first term in an arithmetic progression. Further, according to him, any portion of the series chosen off from the beginning is *iṣṭa* and the rest of the series is *śeṣa,* which forms the remainder series. The sum of those *śeṣa* terms is called *vyutkalita* – subtraction.

Again in श्रेढीबद्धसङ्कलितम्, in the chapter मिश्रकव्यवहार (mixed operations), he has given the formula to find the sum of the series in arithmetic progression in which the common difference is either positive or negative. His formula हीनाधिकचयसङ्कलितधनानयनसूत्रम् is as follows:

व्येकार्धपदोनाधिकचयघातोनान्वितः पुनः प्रभवः।
गच्छाभ्यस्तो हीनाधिकचयसमुदायसङ्कलितम् ।।
– VII.290

The first term is either decreased or increased by the product of the negative or the positive common difference and the quantity obtained by halving the number of terms in the series as diminished by one. Then this is multiplied by the number of terms in arithmetic progression to get the sum of all terms.

So his formula is $S = [a \pm (n - 1)/2 \times d] \times n$.

Mahāvīrācārya discusses series involving fractional terms, fractional common difference, and fractional number of terms too.

RULES GIVEN IN LĪLĀVATĪ

In the *Līlāvatī*, Bhāskara II gives similar rules to find the last term, the mean and the sum.

> व्येकपदघ्नचयो मुखयुक् स्यादन्त्यधनं मुखयुग्दलितं तत्।
> मध्यधनं पदसङ्गुणितं तत्सर्वधनं गणितं च तदुक्तम् ।।
> – v. 121

The common difference multiplied by the period less one, and added to the first term, is the last term. That, added to the first and halved, is the amount of the mean. That multiplied by the period is the sum of the finite arithmetic progression.

ĀRYABHAṬA'S SECOND RULE

The second verse in the *Āryabhaṭīya* gives a complicated formula for finding the number of terms when the sum, first term and common difference are known.

> गच्छोऽष्टोत्तरगुणिताद् दिगुणाद्युत्तरविशेषवर्गयुतात्।
> मूलं द्विगुणाद्यूनं चोत्तरभजितं सरूपार्धम्।। – v. 20

Here the steps are as follows:

Multiply the sum by eight and by the common difference ($8S \times d$).

Increase that by the square of the difference between twice the first term and the common difference [$8Sd + (2a - d)^2$].

Take the square root of that $\sqrt{8Sd + (2a-d)^2}$.

Subtract twice the first term $\sqrt{8Sd + (2a-d)2 - 2a}$.

Divide by the common difference $\left[\sqrt{8Sd + (2a-d)2 - 2a}\right] / d$.

Add one and finally divide by two

$\left\{\left[\sqrt{8Sd + (2a-d)2 - 2a}\right] / d + 1\right\} / 2$.

This gives the formula as if $S = a + (a + d) + (a + 2d) + \ldots$ to n terms, then

$$n = \frac{1}{2}\left[\frac{\sqrt{8dS + (2a-d)^2} - 2a}{d} + 1\right]$$

W.E. Clark, in his translation of the *Āryabhaṭīya*, quotes the remark of Rodet, who translated and published the translation of Gaṇitapāda in the *Journal Asiatique* in 1879, as:

> The development of this formula from the one preceding rule seems to indicate knowledge of the solution of quadratic equation in the form $ax^2 + bx + c = 0$.

All other mathematicians too give sub-rules and numerical problems to calculate one unknown quantity from rest of the three given quantities.

Śreḍhīkṣetram in Sanskrit Texts

Āryabhaṭa, Mahāvīrācārya and Bhāskara II discuss series only in terms of sequences of numbers. But rules given by Śrīdharācārya and Nārāyaṇa Paṇḍita are remarkable as they have given geometrical interpretation of an arithmetic progression. That is why we come across series with fractional periods, negative periods, negative sums and sums equal to zero in their texts. Before discussing the rules in these two texts, it is necessary to know about the concept *śreḍhīkṣetram* in Sanskrit mathematical texts.

Śreḍhī-vyavahāra was interpreted geometrically by some Indian mathematicians. Bhāskara I, in his commentary on the *Āryabhaṭīya*, defines mathematics as:

> गणितं द्विप्रकारम् – राशिगणितम् क्षेत्रगणितम्।

Further, he states that progression and shadow come under geometry:

> अनुपातकुट्टाकारादयो गणितविशेषाः राशिगणितेऽभिहिताः श्रेढीच्छायादयः क्षेत्रगणिते।

Pṛthūdakasvāmī has referred to Skandasena in *Brāhmasphuṭasiddhānta* XII.2 as:

> यच्च स्कन्दसेनाचार्येण श्रेढीन्यायेन सङ्कलितं प्रदर्शितं तत् सङ्कलनं क्षेत्रप्रदर्शनाय।

But we do not find explanation by them why progression comes under geometry. Śrīdharācārya and Nārāyaṇa Paṇḍita are the only two authors, who have applied the method of diagrammatic representation to problems connected with arithmetic progression in their works *Pāṭīgaṇita* and *Gaṇita-Kaumudī* respectively. Both the texts discuss in detail *śreḍhīkṣetram* (series figures). They are plane figures resembling a trapezium with equal flank sides or in some cases they are made of two triangles.

RULES GIVEN IN PĀTĪGAṆITA AND GAṆITA-KAUMUDĪ

Śrīdharācārya and Nārāyaṇa Paṇḍita have considered arithmetic progression as a sequence of numbers as well as in the form of a geometric figure. So, Nārāyaṇa Paṇḍita has discussed progressions in two separate chapters in the *Gaṇita-Kaumudī.* In the chapter titled Śreḍhī-vyavahāra, considering the arithmetic progression as a sequence of numbers, he gives the usual rules for obtaining the last term, mean and the sum of all the terms of a finite arithmetic progression.

व्येकपदघ्नचयो मुखयुक्तोऽन्त्यधनं तु तत्पुनः सादि।
दलितं मध्यधनं तत् पदगुणितं जायते गणितम्।।

व्येकपदार्धघ्नचयः सादिः पदसङ्गुणः भवेद्गणितम्।
– *गणितकौमुदी*, pt. I, p. 105

Śrīdharācārya too gives similar rule to arrive at the sum as:

व्येकपदार्धघ्नचयः सादिः पदसङ्गुणो भवेद् गणितम्।
– *पाटीगणित* v. 85

ŚRĪDHARĀCĀRYA'S RULES IN THE CONTEXT OF ŚREḌHĪKṢETRAM

The second line of the above verse in the *Pātīgaṇita* is:

श्रेढीक्षेत्रे तु फलं भूमुखयोगार्धलम्बहतिः।। – v. 85

Here, the area of the corresponding series figure is given as the area of an isosceles trapezium by the formula: Area = (base + face)/2 × altitude.

Śrīdharācārya, in the beginning of the chapter on series in the *Pāṭīgaṇita*, says:

विस्तारोऽल्पोऽधस्तादुपरि महान् स्याद् यथा शरावस्य।
श्रेढीक्षेत्रस्य तथा गच्छसमो लम्बकस्तस्य।।
– *पाटीगणित* v. 79

According to him, as in the case of an earthen drinking pot, the width at the base is smaller and at the top greater, so also is the case with a series figure. The altitude of that series figure is equal to the number of terms of the series.

Śrīdharācārya gives the details of constructing the series figure in the following verses:

लम्बककरे पृथक् पृथगिष्टादिचयेन तत्फलं भवति।
पदमेकं तल्लम्बश्चयदलहीनं मुखं धरा भवति। सचया सा स्याद् वक्त्रं।

– *पाटीगणित* v. 80-81

The partial areas of the series figure for the successive cubits of the altitude form a series which begins with the given first term and increase successively by the given common difference of the series. The number of terms, say one, is the altitude of the corresponding series figure; the first term of the series as diminished by half the common difference of the series is the base; and that base increased by the common difference of the series is the face.

Śrīdharācārya represents arithmetic progression in the form of an isosceles trapezium narrower at the base and wider at the top with the partial areas as shown in *fig*. 15.1. It should be noted that Śrīdharācārya has taken $(a - d/2)$ as the base of the figure. He constructs the series figure for unit altitude and calls it as *hastikā-kṣetra*, because the unit is *hasta*. According to his rule, base $= (a - d/2)$ and face $= (a - d/2) + d = a + d/2$. Then the face of the actual series figure is calculated using the principle of proportional increase.

Further, he gives the rule as:

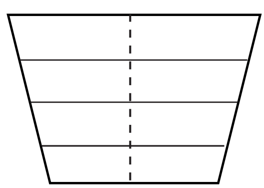


fig. 15.1 Arithimetic progression in the form of an Isosceles

इत्थं श्रेढीक्षेत्रं कृत्वेष्टलम्बके मुखं कल्प्यम् ।
इष्टावलम्बगुणितं धरोनमुखमवनियुग्वदनम ।।

– *पाटीगणित* v. 84

Having constructed the series figure in this manner, one should determine the face for the desired altitude by the following rule:

If the altitude is assumed as unity, then the face minus the base multiplied by the desired altitude and then increased by the base gives the face for the desired altitude.

So the face for altitude n = (face – base) n + face

$= [(a + d/2) - (a - d/2)]\, n + (a - d/2)$

$= d \times n + a - d/2 = a + (n - ½)\, d.$

According to the above rules, base = $(a - d/2)$, face = $a + (n - ½)$ d and altitude = n.

But area = (base + face)/2 × altitude

$= ½\,[(a - d/2) + a + (n - ½)\, d] \times n$

$= ½[a - d/2 + a + nd - d/2] \times n$

$= ½[2a + (n - 1) \times d] \times n$

$= n/2[2a + (n - 1) \times d].$

NĀRĀYAṆA PAṆḌITA'S RULES IN THE CONTEXT OF ŚREDHĪKṢETRAM

Nārāyaṇa Paṇḍita's isosceles trapezium is with wider base and narrower top (*fig*. 15.2). Nārāyaṇa Paṇḍita, in the context of *śreḍhīkṣetram*, gives the first rule in the *Gaṇita-Kaumudī* as:

आदिश्चयदलहीनो वदनं पदचयवधः सवदनो भूः।
गच्छो लम्बो गणितं श्रेढीगणितेन तुल्यं स्यात्।।

– v. 73

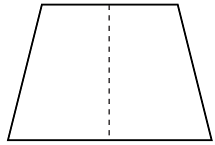


fig. 15.2: Isosceles trapezium with wide base and narrower top

The first term diminished by half the common difference is the face, the product of the period and the common difference increased by the face is the base; the period is the altitude and the area is the sum of the series.

Here the face is $(a - d/2)$ and the base is $(n \times d + a - d/2)$.

Nārāyaṇa Paṇḍita's second verse in the *Gaṇita-Kaumudī* gives the method for calculating the base at any intermediate position on the altitude, i.e. when the altitude is any fractional part of the whole altitude:

अवलम्बखण्डगुणितश्चयः स्ववदनेन संयुतस्तद्भूः।
– v. 74

The fraction of the altitude multiplied by the common difference and combined with its own face is the base (of any segment of the trapezium).

But area = (base + face)/2 × altitude

$$= n/2\,[(n \times d + a - d/2) + (a - d/2)]$$

$$= n/2[n \times d + 2a - d] = n/2[2a + (n - 1) \times d]$$

SERIES FIGURE IN THE FORM OF TWO TRIANGLES

The following verse in the *Gaṇita-Kaumudī* gives the rule to get the sum if the face is negative:

अधरोत्तरे भवेतां त्र्यस्रे भूवदन–भूमिके स्वर्णे।
विवदनकुहृतेकुमुखे लम्बघ्नौ त्र्यस्रयोर्लम्बौ।।

तद्गुणितयोश्च विवरं श्रेढीगणितेन वा तुल्यम्।
– Pt. II, vv. 75-76½

According to Nārāyaṇa Paṇḍita:

ऋणगे वदने तु मिथो भुजौ समाक्रम्य वर्धेते।।
– v. 74

So when the face $(a - d/2)$ is negative the two flanks will cross each other and grow. The figure shows two triangles one positive and one negative with the base and the face as the bases.

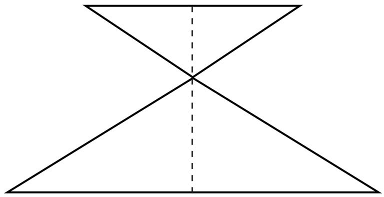


fig. 15.3

According to Śrīdharācārya, when the base $(a - d/2)$ is negative the series figure reduces to two triangles situated one over the other. Here $d/2 > a$. Śrīdharācārya's figure is as given below:

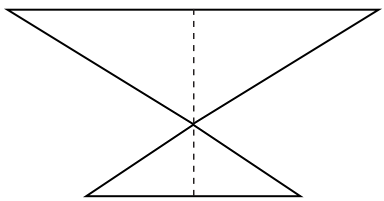


fig. 15.4

His rule to get the altitudes in this case is as follows:

उपरि त्र्यश्रे लम्बो भूमितिरहितेन भाजितं वदनम्।
रूपात्तस्यापगमेऽधस्त्र्यश्रे जायते लम्बः।।

– *पाटीगणित* v. 83

Here the base is negative and the face is positive.

So altitude of the upper triangle

$$= \frac{face \times period}{(base - face)} \times \text{whole altitude.}$$

And altitude of the lower triangle

$$= \frac{base \times period}{(base - face)} \times \text{whole altitude.}$$

The rule given in the *Gaṇita-Kaumudī* is same but there the face is negative and the base is positive. In each case the difference of the areas of the triangles will be equal to the sum of the series. As illustrations both the texts give numerical problems having positive face, negative face, zero face, fractional period and even negative period.

It is noteworthy that Nīlakaṇṭha, in his commentary explains Āryabhaṭa's second rule by constructing *śreḍhīkṣetram*. Also the *Kriyākramakarī* commentary on the *Līlāvatī*, demonstrates similar formula given by Bhāskara II, geometrically by using *śreḍhīkṣetram*.

Numerical Problems

Sanskrit texts give a number of numerical problems to be solved without geometrical figures simply applying the formulae. The distinguishing feature of Sanskrit texts is the rich variety of problems related to the traditions and social customs of that time. Here are some examples:

पञ्चभिराद्यः शङ्खः पञ्चोनशतेन यो भवेदन्त्यम्।
एकादशशङ्खानां यत्तन्मूल्यं त्वमाचक्ष्व।।
— *भास्करकृत आर्यभटीयभाष्य*

The prices of 11 conch shells are in arithmetic progression. The price of the smallest conch shell is Rs. 5 and that of the largest one is Rs. 95. What is the total price of all the 11 conch shells?

केनापि गृहजामातुः षोडशाऽऽद्ये दिने पणाः
प्रदत्ताः पुण्यपुष्यार्थं द्विहान्या च ततः क्रमात्।
दिवसे नवमे जाते कियन्तस्तस्य ते पणाः
संपीड्यैतत्समाचक्ष्व यदि श्रेढ्यां श्रमः कृतः।।
— *ब्राह्मस्फुटसिद्धान्त*

As a form of good act, a man gifted an amount of 16 *paṇa* to his son-in-law on the first day. Then he went on reducing the gift amount by 2 *paṇa* on each successive day. If you have taken pains in learning arithmetic progression, calculate and tell the total amount gifted in 9 days.

त्र्याद्येकोत्तरवृद्ध्या यात्येकः प्रतिदिनं नरस्त्वन्यः।
दश योजनानि कियता कालेन तयोर्गतिस्तुल्या।।
— *पाटीगणित*

One man travels with an initial speed of 3 *yojana*s per day and accelerates by 1 *yojana* per day. Another man travels with the constant speed of 10 *yojana*s per day. In how many days will they cover equal distance?

द्व्यादिना त्रिचयेनाऽऽशु दिनैः षड्भिः समर्जितम्।
वणिजा केनचिन्मध्यमन्त्यं च गणितं वद।।
— *गणितकौमुदी*

A merchant earned 2 units of money on the first day, increased his earning by 3 on each successive day and thus earned money for 6 days in all. Quickly tell his average earnings, final day earnings and total earnings.

द्विकृतिर्मुखं चयोऽष्टौ नगरसहस्रे समर्चितं गणितम्।
गणिताब्धिसमुत्तरणे बाहुबलिन् त्वं समाचक्ष्व।।
— *गणितसारसंग्रह*

The offerings to the gods were made in 1,000 cities, commencing with 4 and increasing the amount by 8 successively. Oh learner! who has enough strength of arms to cross over the ocean of arithmetic, speak out the total value of the offerings.

प्रथमदिने सार्धे द्वे रूपार्धचयेन चान्यदिवसेषु।
वित्तं प्रयच्छति धनी केभ्यः किं सैकमासेन।।
— *त्रिशतिका*

A wealthy man donates two and half rupees on the first day to some people and increases the amount by ½ rupee on each subsequent day. Find the total amount donated by him in one month (1 month = 30 days).

पञ्चाधिकं शतं श्रेढीफलं सप्त पदं किल।
चयं त्रयं वयं विद्मो वदनं वद नंदन।।
— *लीलावती*

It is known that the sum of all terms of an arithmetic progression is 105, number of terms is 7 and common difference is 3. Then, Oh child! tell the first term.

The *Pāṭīgaṇita* and *Gaṇita-Kaumudī* texts contain problems on *śreḍhī-kṣetram.* There are a number of arithmetic progression with diagrams of their *średhī-kṣetra*s. Here is one example:

If $a = 1/2$, $d = 3$, $n = 10/3$ find the sum (*Gaṇita-Kaumudī*, v. 62).

Here

$f = a - d/2 = ½ - 3/2 = -1$ (negative),

$b = n \cdot d + f = 10/3 \times 3 - 1 = 9$

So, the figure will be in the form of two inverted triangles joined at their apexes as shown in *fig*. 15.5:

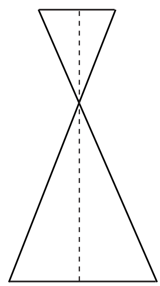


fig. 15.5: Two triangled joined at their apexes

Perpendicular to the base = $(p \times b)/b - f = 30/10 = 3$

Perpendicular to the face = $(p \times f)/b - f = 1/3$

Here area of the bigger triangle = ½ × base × height

= ½ × 9 × 3 = 27/2.

And area of the smaller triangle = ½ × face × height

= ½ × 1 × 1/3 = 1/6.

So the area of the *śreḍhī-kṣetra* = 27/2 – 1/6 = 40/3 which is the sum of the arithmetic progression.

Correlation of Arithmetic Progression with Magic Squares (Magic Squares)

Another important feature of the Sanskrit text *Gaṇita-Kaumudī* is the correlation of arithmetic progression with magic squares. Nārāyaṇa Paṇḍita established relations between terms of a magic square and those of an arithmetic progression. He gives general methods for constructing magic squares along with the principles governing such constructions.

He says:

सर्वेषां भद्राणां श्रेढीरीत्या भवेद् गणितम्।
येषां गणितमभीष्टं साध्यौ तेषां मुखप्रचयौ।।
– XIV.5

All types of magic squares happen to be like the mathematics of series so we have to obtain the first term and the common difference of the magic squares whose sum of all the numbers is desired.

He gives the following definitions. The sum divided by the order is the square's constant. The total number of cells becomes the number of terms of the series and the square root of the former happens to be the *caraṇa* of the square.

He gives a rule to find the first term and the common difference of the arithmetic progression corresponding to a magic squares using the equation:

$-sd + T = na.$

Here

a = the first term,
d = the common difference,
n = number of cells,
s is the sum of the first $(n-1)$ natural numbers, and
T is the total.

In an example, he has asked to find the first term and the common difference of the magic square of order 4 whose square's constant is 40.

Here square's constant is 40, order = 4, so number of terms = 16.

So, $T = 40 \times 4 = 160$ and s = sum of first 15 natural numbers = 120.

Here by solving the equation $-s \times d + T = n \times a$ we have $-15d = 2a - 20$.

So we get two possibilities by solving using *kuṭṭaka:*

1. $a = 10$ and $d = 0$

and

2. $a = -5$ and $d = 2$.

In the first case each cell will contain 10, so another possibility should be considered. Then we have an arithmetic progression – 5, – 3, – 1, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25.

Another square where $a = -14$, $d = 4$, square's constant = 64.

We have an arithmetic progression – 14, – 10, – 6, – 2, 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46.

The arithmetic progressions can be represented as magic squares:

– 5	9	19	17
21	15	– 3	7
1	3	25	11
23	13	– 1	5

– 14	14	34	30
38	26	– 10	10
– 2	2	46	18
42	22	– 6	6

Other beautiful patterns given in the *Gaṇita-Kaumudī* can be formed with the help of magic squares and magic rectangles. There are figures whose cells are divided by their diagonals into two equal parts. Figures in the form of lotus with 8 petals, figures named as *sarvatobhadra*, *padmabhadra* and figures within figures are very attractive.

Concluding Remarks

Mathematics textbooks can be regarded as the most accountable historical proof for the whole mathematics education history. Also textbook is one of the major factors that influences students' learning. The topic of arithmetic progression is currently prescribed for the standard 10 across the three major educational boards in India, viz. SSCE, CBSE and ICSE. This topic has historical significance in mathematics education in India. The fifth-century mathematician Āryabhaṭa was the first Indian mathematician to give properties and rules regarding arithmetic progression in the *Āryabhaṭīya*. The topic has been handled in a variety of ways from educational point of view in other Sanskrit texts too. The method of exposing the concept and the development of solution techniques shows some remarkable differences in ancient and modern texts. Two important points to be noted are geometrical interpretation of arithmetic progression and the relation of an arithmetic progression to magic squares.

Diagrammatic treatment of arithmetical series seems to be a unique feature of Sanskrit texts. This type of geometrical interpretation of an arithmetic progression is not found in modern texts. Geometrical representation would be useful to show a relation between two branches, viz. arithmetic and geometry. Actually multiple representations should be appropriately integrated into textbooks to enhance conceptual understanding of the students. The mention of old Sanskrit texts in this context would arouse students' curiosity in our history and culture too. Even in modern textbooks one of the aims is given as to develop an interest in mathematics. This is possible by recreational mathematics, and magic squares are very important in the field of recreational mathematics. Activities like constructing magic squares should be integrated into the classroom to enable students to better understand mathematics.

The utility of mathematics in daily chores is appreciated by all. Mathematics learning should connect to real world problems. Numerical problems in modern texts are related to daily life activities of modern people. These applications include earning interest on savings, loan repayment instalments, potato race where the distance between the potatoes increases with a common difference, an asset depreciated by a fixed amount per year, building a ladder with sloping sides, etc. The numerical problems given in Sanskrit texts give a flavour of different time in India. It is our cultural heritage. For example, Sanskrit texts give a number of problems about offerings to charities. This shows the importance given to this life value in those days. So some numerical problems of that period can be included in modern texts.

References

Amma, T.A. Saraswati, 1999, *Geometry in Ancient and Medieval India*, 2nd rev. edn, Delhi: Motilal Banarsidass.

Āryabhaṭīya, pt. II, with the commentary of Bhāskara I and Someśvara, critically ed. K.S. Shukla and K.V. Sarma, New Delhi: Indian National Science Academy, 1976.

Bhāskarīyabījagaṇitam, with *Navāṅkura* commentary by Kṛṣṇa Daivajña, ed. V.G. Apte, Ānandāśrama Sanskṛta Granthāvalī 99, 1930.

Brāhmasphuṭasiddhānta, with *Vāsanā*, *Vijñāna* and Hindi commentaries, vols I-IV, ed. A board of editors headed by R.S. Sharma, New Delhi: Indian Institute of Astronomical and Sanskrit Research, 1966.

Colebrooke, H.T., 2005, *Classics of Indian Mathematics, Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhāskara*, Delhi: Sharada Publishing House.

Gaṇita-Kaumudī of Nārāyaṇa Paṇḍita, part II, ed. Padmakara Dvivedi, Benares: Government Sanskrit College, 1942.

Gaṇita-Kaumudī of Nārāyaṇa Paṇḍita, English tr. with notes by Parmanand Singh, article published in *Gaṇita Bhāratī*, 1998, 20(1-4): 25-82, Delhi: Bulletin of Indian Society of History of Mathematics.

Gaṇitasāra-Saṁgraha of Mahāvīrācārya, ed. with tr. and notes by M. Raṅgācārya, Madras Government Publication, 1912.

Pāṭīgaṇita of Śrīdharācārya with an ancient Sanskrit commentary, ed. and tr. Kripa Shankar Shukla, Department of Mathematics and Astronomy, Lucknow University, Lucknow, 1959.

Plofker Kim, 2009, *Mathematics in India: 500 BCE – 1800 CE*, Princeton, NJ: Princeton University Press.

The Āryabhaṭīya of Āryabhaṭa, tr. with notes by W.E. Clark, Chicago: The University of Chicago Press, 1930 (recomposed edn, D.K. Printworld, New Delhi, 2016).

Triśatikā of Śrīdharācārya, ed. Sudhakara Dvivedi, printed and published by Pandit Jagannatha Sarma Mehta, Benares, 1899.

16

Contributions of Shri Bapu Deva Shastri to Līlāvatī of Bhāskarācārya

B.Vijayalakshmi

IN THE texts on the history of mathematics in ancient India, the names of Āryabhaṭa and Bhāskara occupy prominent position. The *Līlāvatī* of Bhāskara is perhaps the most read book on *vyaktagaṇita* (arithmetic) in ancient India, as well as in modern times, and this book has been translated and critically edited by various mathematicians in different languages all over the world. Bapu Deva Shastri (BDS), a reputed mathematician of the nineteenth century, has critically edited the *Līlavatī* in Sanskrit in his book *Līlāvatī: Treatise on Arithmetic by Bhāskarācārya* (Benares 1883) with the explanations inclusive of new examples, new *sūtra*s and *upapatti*s.

Born in Pune, in the year 1821, Nrisimha Shastri alias Bapu Deva Shastri got his education in algebra and arithmetic in Nagpur. Lancelot Wilkinson recognized his talents in Gaṇitaśāstra and helped him getting a post in a reputed *pāṭhaśālā* in Kāśī. He was a keen and enthusiastic scholar of ancient Indian mathematics. He has brought out books related to Indian mathematics, which include translation of the Sanskrit works such as the *Sūrya-Siddhānta, Siddhānta-Śiromaṇi* and *Līlāvatī* of Bhāskarācārya into English. He has conducted classes for teachers and research scholars; presented papers at various university's national and international seminars. He has made certain value addition in the topics of Division, Square, Supposition, Pulverization, Progression etc. for the benefit of better understanding of the students. In the present paper, I wish to throw light on some of his techniques and examples as detailed in the book mentioned above.

1 Chapter on Division

The *sūtra* given is:

> भाज्याद्धरः शुद्ध्यति यद्गुणः स्यादन्त्यात्फलं तत्खलु भागहारे।
> समेन केनाप्यपवर्त्य हारभाज्यौ भवेद्वा सति संभवे तु।।
>
> *bhājyāddharaḥ śuddhyati yadguṇaḥ syādantyātphalaṁ tatkhalu bhāgahāre*।
> *samena kenāpyapavartya hārabhājyau bhavedvā sati saṁbhave tu*।। – *Līlāvatī* XIX

Find the largest integer whose product with the divisor can be subtracted from the extreme left-hand side digit(s) of the dividend. This integer is the first digit of the quotient. If the divisor and the dividend have a common factor, then the common factor can be cancelled and the division is carried out with the remaining factors.

When the divisor has more digits, the method given by Bapu Deva Shastri is as detailed below:

> यदा भाजकोऽनेकाङ्कविशिष्टः स्यात, तदा लघुर्भजन प्रकारः। लब्धिः अङ्कगुणितभाजकं अन्त्यगुणित भाज्याधो न लिखेत्। किन्तु तम् अन्त्य भाज्याद्विशोध्य शेषं न्यसेत्। तच्छेषावगमक प्रकारोऽयम् ।
>
> *yadā bhājako 'nekāṅkaviśiṣṭaḥ syāt, tadā laghurbhajana prakāraḥ*। *labdhiḥ aṅkaguṇitabhājakaṁ antyaguṇita bhajyādho na likhet*। *kintu tam antya bhājyādviśodhya śeṣaṁ nyaset*। *taccheṣāvagamaka prakāro 'yam*। – *Līlāvatī (BDS)*, p. 5

Do not write the product of the divisor and quotient under the dividend, but subtract it from the dividend and write the remainder under the dividend.

It is to be noted that this method of division is similar to the Hindu method of performing operations on a *pātī.* In the *pātī* method, here is an example: 1620/12.

Stage	*No./Divisor*	*Line of Quotient*
First	1620 12	
Second	420 12	1
Third	420 12	1
Fourth	60 12	13
Fifth	60 12	13
Sixth	0	135

The example given by Bapu Deva Shastri is as follows:

भाजकः	भाज्यः	लब्धिः	शेषम्
Divisor	Dividend	Quotient	Reminder
5231)	354269831	(67725	356
	40409	1	
	37928	2	
	13113	3	
	26511	4	
	356.	5	

Now 1 2 3 4 5 are calculated as follows:

```
Divisor  Dividend  Quotient     Remainder
5231) 354269831 (67725            356
      31386
      -----
       40409     1
       36617
       -----
        37928    2
        36617
        -----
         13113   3
         10462
         -----
          26511  4
          26155
            356. 5
```

Here Bapu Deva Shastri also gives some rules with respect to the division by numbers, for easy *apavartanam* (abridgement) – reduction of the dividend and divisor.

अत्रापवर्तनस्य शीघ्रमुपस्थितये कतिचन सङ्ख्या विशेषधर्माः प्रदर्श्यन्ते ।

atrāpavartanasya śīghramupasthitaye katican saṁkhyā viśeṣadharmāḥ pradarśyante । – *Līlāvatī* (BDS), p. 6

These are the rules to test for divisibility by 2, 3, 4 …. 11.

Finding Square of a Number

The *sūtra* given is:

समद्विघातः कृतिरुच्यतेऽथ स्थाप्योन्त्यवर्गो द्विगुणान्त्यनिघ्नाः।
स्वस्वोपरिष्ठाच्च तथापरेऽङ्कास्त्यक्त्वान्त्यमुत्सार्य पुनश्चराशिम्।।

samadvidhātaḥ kṛtirucyate 'tha sthāpyontyavargo viguṇāntya-nighnāḥ ।।
svasvopariṣṭhācca tathāpare 'ṅkāstyaktvāntyamutsārya punaścarāśim ।।
– *Līlāvatī* XX

The product of a number with itself is called a square. To square a number, use the following procedure. First write the square of the extreme left-hand digit on its top. Then multiply the next (i.e. second) digit by the double of the first digit and write the result on the top. Next multiply the third digit by the double of the first digit and write the result on the top. In this way arrive at the unit's place. Next cross the first digit and shift the number so formed one place to the right. Then repeat the same procedure. Finally add all the products written at the top and the sum is the required answer.

The method suggested by Bapu Deva Shastri is as follows:

वर्गार्थं कार्यं आद्यङ्कतो वा अन्त्यङ्कतो वा समानम्।

vargārthaṁ kāryaṁ ādyaṅkato vā antyaṅkato vā samānam ।
– *Līlāvatī* (BDS), p. 7

The process of squaring may start from the first digit or the last digit:

1. Multiply the first digit with itself and then with twice the first digit multiply the digits to the left of it and add both. This is the first part of the result.
2. Multiply the second digit with itself and then with twice the second digit multiply the digits to the left of it and add both. This is the second part of the result and continue this.

Now write all these results in the order one below the other so that in the 100th place (*sthāna*) of the previous result the first digit of the successive result is written, when all these are added we get the required square.

The formula is,

$$\begin{aligned}(f+e+d+c+b+a)^2 = a^2 &+ 2a(\mathrm{f}+\mathrm{e}+d+c+b)\\ &+ b^2 + 2b(f+e+d+c)\\ &+ c^2 + 2c(f+e+d)\\ &+ d^2 + 2d(f+e)\\ &+ e^2 + 2fe\\ &+ f^2\end{aligned}$$

The example given by Bapu Deva Shastri is as follows:

Find the square of 547913.

547913		
3287469	1	First part
109581	2	Second part
98541	3	
7609	4	
416	5	
25	6	
300208655569		

The ciphers are omitted for simplicity.

This is the square value. The calculations for 1 2 3 4 5 6 are as follows:

1. $3^2 + 2 \times 3 \times (54791) \times 10 = 9 + 6 \times (54791) \times 10 = 9 + 3287460$
$= 3287469.$

2. $1^2 \times 10^2 + 2 \times 1 \times (5479) \times 10^3 = 100 + 10958 \times 10^3 = 10958100.$

3. $9^2 \times 10^4 + 2 \times 9 \times (547) \times 10^5 = 810000 + 18 \times (547) \times 10^5$

$= 810000 + 984600000 = 985410000.$

4. $7^2 \times 10^6 + 2 \times 7 \times (54) \times 10^7 = 49000000 + 14 \times 540000000$

$= 49000000 + 7560000000 = 7609000000.$

5. $4^2 \times 10^8 + 2 \times 4 \times (5) \times 10^9 = 16 \times 10^8 + 40 \times 10^9 = (16 + 400) \times 10^8$

$= 41600000000.$

6. $5^2 \times 10^{10} = 250000000000.$

The same can also be done (proved) from last digit (*antya*).

Finding Square Root of a Number

The sūtra given is:

त्यक्त्वान्त्याद्विषमात् कृतिं द्विगुणयेन्मूलं समे तद्धृते
त्यक्त्वा लब्धकृतिं तदाद्यविषमाल्लब्धं द्विनिघ्नं न्यसेत्।
पंक्त्या पंक्तिहृते समेन्त्यविषमात्त्यक्त्वाप्तवर्गं फलम्
पंक्त्यां तद्विगुणं न्यसेदिति मुहुः पंक्तेर्दलं स्यात्पदम्।।

tyāktvāntyādviṣamāt kr̥tiṁ dviguṇayenmūlaṁ same taddhr̥te
tyaktvā labdhakr̥tiṁ tadādyaviṣamāllabthaṁ dvinighnaṁ nyaset।
paṅktyā paṅktihr̥te samentyaviṣamāttyaktvāptavargaṁ phalam
paṅktyāṁ taddviguṇaṁ nyaseditimuhuḥ paṅkterdalaṁ syātpadam।।
– *Līlāvatī* XXIII

Here according to Bapu Deva Shastri *viṣama* consisting of "two" digits is brought down at every stage, for calculation of square root and not one digit as in the ancient method and his working is similar to the one which we are using now (*Līlāvatī* (BDS), pp. 8-9).

Finding Cube Root of a Number

The *sūtra*s given are:

आद्यं घनस्थानमथाघने द्वे पुनस्तथान्त्याद्घनतो विशोध्य ।
घनं पृथक्स्थं पदमस्य कृत्या त्रिघ्न्या तदाद्यं विभजेत् फलं तु ।।

पंक्त्यां न्यसेत्तत्कृतिमन्त्यनिघ्नीं त्रिघ्नीं त्यजेत्तत्प्रथमात्फलस्य ।
घनं तदाद्यात् घनमूलमेवं पंक्तिं भवेदेवमतः पुनश्च ।।

ādhyṁ ghanasthānamathāghane dve punastathāntyādghanato viśodhya ।

ghanaṁ pṛthaksthaṁ padamasya kṛtyā trighnyā tadādyaṁ vibhajet phalaṁ tu ।। – *Līlāvatī* XXIX-XXX

paṅktyāṁ nyasettatkṛtimantyanighnīṁ trighnīṁ tyajettat prathamātphalasya ।
ghanaṁ tadādyāt ghanamūlamevaṁ paṅktiṁ bhavedevamataḥ punaśca ।।

Here, according to Bapu Deva Shastri *ghana* consisting of "three digits" is brought down at every stage, for calculation of cube root and not one digit for calculation of cube root and his working is simpler than the one given by others and similar to the one which we are doing now (*Līlāvatī* (BDS), pp. 10-11**).**

Below we work out the cube root of 817400375 as given by Bapu Deva Shastri:

			घनः	घनमूलः
			...	
			817400375	(935
पङ्क्तिः (*paniktiḥ*)			729a^3	
273$_3a+b$	अपूर्णभाजकः (*apūrṇabhājaka*)	24300$_3a^2$	88400	
62b	क्षेपः (*kṣepaḥ*)	819 $b \times (3a+b)$		
2795$_3a+b+2b+c$	पूर्णभाजकः (*pūrṇabhājakaḥ*) 9(b^2)	25119 $3a^2+b \times (3a+b)$		75357
	अपूर्णभाजकः (*apūrṇabhājakaḥ*)	2594700b $(3a+b)+3a^2$ $+b(3a+b)+b^2$		13043375
	क्षेपः (*kṣepaḥ*)	13975 $c\,[(3a+b)+2b+c]$		

पूर्णभाजकः (*pūrṇabhājakaḥ*) 2608675. $b^2 + b(3a + b) +$ $3a^2 + b(3a + b) + b^2 + c[(3a + b) + 2b + c]$
13043375

The calculations of पङ्क्ति (*paṅktiḥ*):, अपूर्णभाजकः (*apūrṇabhājakaḥ*) पूर्णभाजकः (*pūrṇabhājakaḥ*), क्षेपः (*kṣepaḥ*) are explained in detail by Bapu Deva Shastri as follows:

THE PRACTICAL PART

In the problem detailed above, the first digit in the quotient is 9 = a. Write $3a = 27$ in पङ्क्तिः (*paṅktiḥ*), write $3a^2 \times 100 = 24300$ as अपूर्णभाजकः (*apūrṇabhājakaḥ*). The next possible digit in quotient is 3 and so we write 3 in quotient as well as पङ्क्तिः (*paṅktiḥ*) in proper place. Now multiply पङ्क्तिः (*paṅktiḥ*) with the recent digit in root and write this as क्षेपः (*kṣepaḥ*). Then the पूर्णभाजकः (*pūrṇabhājakaḥ*) is sum of अपूर्णभाजकः (*apūrṇabhājakaḥ*) and क्षेपः (*kṣepaḥ*).

Then multiply the पूर्णभाजकः (*pūrṇabhājakaḥ*) with recent digit in root and subtract it from dividend and then the next *ghana* is brought down. Then the square of recent digit in root is written below पूर्णभाजकः (*pūrṇabhājakaḥ*) and the next अपूर्णभाजकः (*apūrṇabhājakaḥ*) is calculated as sum of previous पूर्णभाजकः (*pūrṇabhājakaḥ*), क्षेपः (*kṣepaḥ*) and the square now written. Also write 2 × recent digit in root in पङ्क्तिः (*paṅktiḥ*) in proper place and add. And now the procedure is repeated.

THE THEORY PART

$$\begin{aligned}(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3\\ &= a^3 + b(3a^2 + 3ab + b^2)\\ &= a^3 + b\{3a^2 + b(3a + b)\} \qquad (1)\end{aligned}$$

Similarly,

$$(a + b + c)^3 = (a + b)^3 + c\{3(a + b)^2 + c\ (3(a + b) + c)\} \qquad (2)$$

$$\{3(a + b)^2 + c(3(a + b) + c)\} = 3a^2 + 3b^2 + 6ab + 3ac + 3bc + c^2 \qquad (3)$$

Again,

$$b(3a + b) + 3a^2 + b(3a + b) + b^2 + c[(3a + b) + 2b + c]$$

$$= 3ab + b^2 + 3a^2 + 3ab + b^2 + b^2 + 3ac + 3bc + 2bc \qquad (4)$$

$(3) = (4)$.

So,

$$\{3(a+b)^2 + c(3(a+b)+c)\}$$
$$= b(3a+b) + 3a^2 + b(3a+b) + b^2 + c[(3a+b) + 2b + c].$$

Finding the Unknown Quantity (Subject to Certain Conditions)

The *sūtra* given is:

उद्देशकालापवदिष्टराशिः क्षुण्णोहृतोंशौ रहितो युतोवा ।
इष्टाहतं दृष्टमनेन भक्तं राशिर्भवेत् प्रोक्तमितीष्टकर्म ।।
uddeśakālāpavadiṣṭarāśiḥ kṣuṇṇohṛtoṁśau rahito yutovā।
iṣṭāhataṁ dṛṣṭamanena bhaktaṁ rāśirbhavet proktamitīṣṭakarma।।

– *Līlāvatī* LII

This method is also known as supposition operation with an assumed number. It is the rule of false position, supposition and trial and error. To discover the unknown number, begin with any convenient number. Then according to the conditions given in the problem, carry on the operations such as multiplication and division. [Then the given quantity, being multiplied by the assumed number and divided by that (which has been found), yields the number sought. This is called the process of supposition.]

It is really very interesting to note how problems on *avyakta gaṇita* (algebra) were solved with *vyakta gaṇita* (arithmetic) with this method of supposition. Normally, these types of problems are solved by assuming the unknown to be one and then proceeding with the other operations and from दृष्ट (*dṛṣṭa*) we find इष्ट (*iṣṭa*).

Here Bapu Deva Shastri uses त्रैराशिकं (*trairāśikaṁ*) and इष्ट (*iṣṭa*) and gives two examples to solve one unknown and two unknowns. The method according to Bapu Deva Shastri can be better explained, in the following manner:

If we have to find an unknown value, say U, with a given condition:

Suppose its value to be s_1. Apply the condition. It may not satisfy the condition. Find the difference d_1. Suppose its value to be s_2. Apply the condition. It may not satisfy the condition. Find the

> difference d_2, now find the difference between d_1 and d_2. Also, the difference between s_1d_2 and s_2d_1. Now $(s_1d_2 \sim s_2d_1)/(d_1 \sim d_2)$ gives the unknown. – *Līlāvatī* (BDS), pp. 19-20

This principle has been used in the following two examples by Bapu Deva Shastri.

Example 1: Finding the price of a horse

> एकस्य रूपत्रिशती षडश्वा अश्वा दशान्यस्यतु तुल्यमूल्याः ।
> ऋणं तथा रूपशतं च तस्य तौ तुल्यवित्तौ च किमश्वमूल्यम् ।।
>
> यदाद्य वित्तस्य दलं द्वियुक्तम् तत्तुल्यवित्तो यदिवाद्वितीयः ।
> आद्यो धनेन त्रिगुणोन्यतो वा पृथक् पृथङ्मे वद वाजि मूल्यम् ।।
>
> *ekasya rūpatriśati ṣaḍaśvā aśvā daśānyasyatu tulyamūlyāḥ*।
> *r̥ṇaṁ tathā rūpaśataṁ ca tasya tau tulyavittau ca śvamūlyam*।।
>
> *yadādya vittasya dalaṁ dviyuktam tattulyavitto adivādvitīyaḥ*।
> *ādyo dhanena triguṇonyato vā pr̥thak pr̥thaṅme vada vāji mūlyam*।।
>
> – *Līlāvatī* (BDS), p. 20

There are totally three problems given in the example and will be considered one after another.

In all these problems, we assume only one value.

Problem (a): Condition given in first two lines. Two persons have 6 horses and 10 horses each. The first person has Rs. 300 and the second person has a debt of Rs. 100 but the total value of horses and money for both persons are same.

We will assume 50 to be the price of horse.

Then first person has $50 \times 6 + 300 = 600$. (1)

Second person has $50 \times 10 - 100 = 400$. (2)

(1) and (2) are not equal, difference is + 200

We will assume 80 to be the price of horse.

Then first person has $80 \times 6 + 300 = 780$. (3)

Second person has $80 \times 10 - 100 = 700$. (4)

(3) and (4) are not equal, difference is + 80.

Assuming 50 the difference is + 200.

Assuming 80 the difference is + 80.

We multiply each difference with the other supposition and get their difference

I.e. $200 \times 80 - 80 \times 50 = 16000 - 4000 = 12000$.

Difference between actual differences $200 - 80 = 120$.

Hence the value of a horse is $12000/120 = 100$.

Problem (*b*): Here the conditions are as follows:

Half the money value of the first added with 2 is equal to the second person's money value.

Proceeding as above we get the value of a horse: 36.

Problem (*c*): Here the conditions are as follows:

The money value of the first is equal to three times money value of the second person.

Proceeding as above we get the value of a horse: 25.

Next, we see another example given by Bapu Deva Shastri, finding two values given two conditions.

Example 2

एको ब्रवीति मम देहि शतं धनेन त्वत्तो भवामि हि सखे द्विगुणस्ततोऽन्यः ।
ब्रूते दशार्पयसि चेन्ममषड्गुणोऽहम् त्वत्तस्तयोर्वद धन मम किं प्रमाणे ॥

eko bravīti mama dehi śataṁ dhanena tvatto bhavāmi hi sakhe dviguṇastato ' nyaḥ ।
brūte daśārpayasi eenmamaṣaḍguṇo 'ham tvattastyorvada dhana mama kiṁ pramāṇe॥

– *Līlāvatī* (BDS), p. 20

Two friends are having two different amounts.

Condition 1: The first person says, "if you give me Rs. 100, then the amount in my hand will be twice as much as you have".

Condition 2: The second person says, "if you give me Rs. 10, then the amount in my hand will be six times as much as you have". Find their amounts.

We will do this problem using supposition in two stages.

• *Stage 1*

Part A – Fix the first amount be 20. (First amount >10) now fixed.

We will find the second amount satisfying the first condition:

(a) Assuming the second amount be 110. (Second amount > 100) difference is 100.

(b) Assuming the second amount be 120. (Second amount > 100 difference is 80.

And get the money with the second person as 160.

Hence (20, 160) is one set of value satisfying the first condition.

Part B – Fix the first amount be 100. (First amount >10) now fixed.

We will find the second amount satisfying the first condition.

(a) Assuming the second amount be 150. (Second amount > 100) difference is 100.

(b) Assuming the second amount be 180. (Second amount > 100) difference is 40.

And get the money with the second person less as 200.

Hence (100, 200) is the second set of value satisfying the first condition.

• *Stage 2*

We will now find the set satisfying both conditions.

Assume the first set satisfying the first and second conditions to be (20, 160).

Now we will apply the second condition.

The difference is 100.

Assume the second set satisfying the first and second conditions to be (100, 200).

Now we will apply the second condition.

The difference is 330.

Assuming 20 for the first person, the difference is 110.

Assuming 100 for the first person, the difference is 330.

Hence the money with the first person is 40.

Now, to find the money with the second person:

Assuming 160 for the second person, the difference is 110 (as above).

Assuming 200 for the first person, the difference is 330 (as above).

Hence the money with the second person: 74800/440 = 170.

Money with the first person = 40.

Money with the second person = 170.

Square Transition (Vargakarma)

The *sūtra*s given are:

> इष्टकृतिरष्टगुणिता व्येका दलिता विभाजितेष्टेन ।
> एकः स्यादस्यकृतिर्दलिता सैकापरो राशिः ।।
> रूपं द्विगुणेष्टहृतं सेष्टं प्रथमोऽथवापरो रूपम् ।
> कृतियुतिवियुती व्येके वर्गौ स्यातां ययो राश्योः ।।
>
> *iṣṭakṛtiraṣṭaguṇītā vyekā dalitā vibhājiteṣṭena* ।
> *ekaḥ syādasyakṛtirdalitā saikāparo rāśiḥ* ।।
> *rūpaṁ dviguṇeṣṭahṛtaṁ seṣṭaṁ prathamo 'thavāparo rūpam* ।
> *kṛtiyutiviyuti vyeke vargau syātaṁ yayo rāśyoḥ* ।।
>
> – *Līlāvatī* LXV-VI

A certain problem relating to squares is propounded here.

Here we see an indeterminate problem that admits innumerable solutions. We find two *rāśi*s R_1 and R_2 such that $R_2^2 + R_1^2 - 1$ and $R_2^2 - R_1^2 - 1$ are perfect squares.

Given: $R_1 = b = (8t^2 - 1)/2t$, and $R_2 = b^2/2 + 1$ are the two *rāśi*s. To prove $R_2^2 + R_1^2 - 1$ and $R_2^2 - R_1^2 - 1$ are perfect squares where t is the *iṣṭa*.

The उपपत्ति (*upapatti*) given by Bapu Deva Shastri is as follows (*Līlāvatī* (BDS), p. 22):

The substitution used is $a = b^2/2$. (A)

Suppose the first *rāśi* $R_1 = b$; second *rāśi* $R_2 = a + 1$. (B)

Their squares are b^2 and $a^2 + 2a + 1$.

Choose $2a = b^2$ by (A).

Then we have:

$R_2^2 - R_1^2 - 1 = a^2 + 2a + 1 - b^2 - 1 = a^2 + 2a - b^2 = a^2$, a perfect square by (A).

Hence the first result is proved.

Now substituting (A) in R_2 we get $R_2 = b^2/2 + 1$:

$R_2^2 + R_1^2 - 1 = (b^2/2 + 1)^2 + b^2 - 1 = b^4/4 + b^2 + 1 + b^2 - 1 = b^4/4 + 2b^2 = b^2(b^2/4 + 2)$.

This will be a perfect square provided $(b^2/4 + 2)$ is a perfect square.

Now putting $b = (8t^2 - 1)/2t$. (1)

We get $(b^2/4 + 2) = (8t^2 - 1)^2/16t^2 + 2 = (8t^2 + 1)^2/16t^2$ which is a perfect square.

From (A) we have $a = b^2/2$.

So $R_1 = b = (8t^2 - 1)/2t$ and $R_2 = a + 1 = b^2/2 + 1$ are the two *rāśi*s which satisfy the conditions that $R_2^2 - R_1^2 - 1$, $R_2^2 + R_1^2 - 1$ are squares.

Given: $1/(2x) + x$ and 1 are the two *rāśi*s.

The उपपत्ति (*upapatti*) given by Bapu Deva Shastri is as follows: (*Līlāvatī* (BDS), p. 22):

Suppose the first *rāśi* $R_1 = 1/(2t) + t =$; the second *rāśi* $R_2 = 1$. (B)

Then

$R_2^2 - R_1^2 - 1 = (1/(2t) - t)^2$

$R_2^2 + R_1^2 - 1 = (1/(2t) + t)^2$ and both are squares.

As the two preceding solutions give fractional solutions, the next *sūtra* by *ācārya* is to find answers in whole numbers:

इष्टस्य वर्गवर्गो घनश्चतावष्टसंगुणौ प्रथमः ।
सैको राशी स्यातामेवं व्यक्तेऽथवाऽव्यक्ते ॥

iṣṭasya vargavargo ghanaścatāvṣṭasṁguṇau prathamaḥ ।
saiko rāśī syātāmevaṁ vyakte 'thavā vyakte ॥
– *Līlāvatī* LXVIII

Given: $8x^3$ and $8x^4 + 1$ are the two *rāśi*s.

The उपपत्ति (*upapatti*) given by Bapu Deva Shastri is as follows (*Līlāvatī* (BDS), p. 23):

The substitutions used are $2b = n^2$, $a^2 = 2bn$, $n = 4x^2$

Suppose the first *rāśi* $R_1 = a$; the second *rāśi* $R_2 = b + 1$. (B)

Their squares are a^2 and $b^2 + 2b + 1$.

Then we have

$R_2^2 - R_1^2 - 1 = b^2 + 2b + 1 - a^2 - 1 = b^2 + 2b - a^2$.

$R_2^2 + R_1^2 - 1 = b^2 + 2b + a^2$.

The above two must be perfect squares.

So, we put $2b = n^2$, $a^2 = 2bn$ which make the above two perfect squares.

Thus $a^2 = 2bn = n^3$. Now put $n = 4x^2$

Then $a^2 = 64\, x^6$,

The first *rāśi* $R_1 = a = 8x^3$.

In the same way $b = n^2/2 = 8x^4$.

The second *rāśi* $R_2 = b + 1 = 8x^4 + 1$.

Next Bapu Deva Shastri refutes that when *iṣṭa* < 2 or *iṣṭa* < 1/2 the *sūtra*s quoted by Bālakr̥ṣṇa Daivajña and Lakṣmīdāsa, *viyuti pakṣe* does not hold (*Līlāvatī* (BDS), p. 23).

The first *sūtra* by Bālakr̥ṣṇa is:

इष्टः प्रथमो राशिर्निजार्धनिहतः स एवान्यः ।
अनयोः कृतियुतिवियुती रूपयुते मूलदे स्याताम् ॥

iṣṭaḥ prathamo rāśirnijārdhanihataḥ sa evānyaḥ ।
anayoḥ kr̥tiyutiviyutī rūpayute mūlade syātām ॥

The first *rāśi* = x.

The second *rāśi* = $x^2/2$.

The first sum to be calculated is

$x^2 + (x^2/2)^2 + 1 = (x^4 + 4x^2 + 4)/4$ is perfect square of $(x^2 + 2)/2$ always.

The second one is

x^2 diff $(x^2/2)^2 + 1 = (x^4 - 4x^2 + 4)/4$ square of $(x^2 - 2)/2$ when $x^4 > 4x^2$,

i.e. when $x^4/4x^2 > 1$ or when $x^4/4x^2 = 1$,

i.e. when $x^2/4 > 1$ or when $x^2/4 = 1$,

i.e. when $x/2 > 1$ or when $x/2 = 1$,

i.e. when $x > 2$ or when $x = 2$.

Hence, when *iṣṭa* < 2, the *sūtra* quoted by Bālakṛṣṇa *viyuti pakṣe* does not hold.

The second *sūtra* by Lakṣmīdāsa is:

चतुर्गुणेष्टमाद्यमः स द्विघ्नोऽभीष्टसंगुणोऽपरे ।
अनयोः कृतियुतिवियुती रूपयुते मूलदे स्याताम् ।।

caturguṇeṣṭamādyamaḥ sa dvighno 'bhīṣṭasaṁguṇo 'pare ।
anayoḥ kṛtiyutiviyutī rūpayute mulade syātām ।।

This one is the same as the above for the two *rāśi*s are $4x$ and $8x^2$.

Same as $4x$ and $(4x)^2/2$.

Same form as x and $x^2/2$.

Hence, when $4x < 2$ or x (*iṣṭa*) $< 1/2$, the *sūtra* quoted by Lakṣmīdāsa *mithaviyuti pakṣe* does not hold.

In the chapter on Mensuration, the *sūtra* to construct a right-angled triangle with given two quantities is as follows (*Līlāvatī* CLIII):

इष्टयोराहतिर्द्विघ्नी कोटिर्वर्गान्तरं भुजः।
कृतियोगस्तयोरेवं कर्णश्चाकरणीगतः ।।

iṣṭayorāhatirdvighnī koṭirvargāntaraṁ bhujaḥ ।
kṛtiyogastayorevaṁ karṇaścākaraṇīgataḥ ।।

If x, y are taken as two *iṣṭa*s,

$x^2 - y^2$, $2xy$, $x^2 + y^2$ are the sides of right-angled triangle.

If x, y (1, 2) are taken as two *iṣṭa*s,

sides of triangle are $x^2 - y^2$, $2xy$, i.e. 3, 4. *Hyp*^{2} is $3^2 + 4^2$ is 5^2, *karṇa* = 5.

Bapu Deva Shastri further extends this to find two perfect squares of the form $x^2 + y^2 - 1$ and $x^2 - y^2 + 1$ (Līlāvatī (BDS), p. 24).

इष्टयोराहतिर्द्विघ्नीत्याद्याचार्योक्त मार्गतः ।
कोटिदोः श्रुतयः साध्यास्तत्रकोटिभुजाहतिः ।।

द्विनिघ्नीपरसंज्ञा स्यादिष्टवर्गपराख्ययोः ।
योगात् तदन्तरेणापितमेको राशिर्भवेत्तथा ।।

तेनान्तरेण हृद्विघ्नेष्टघ्नः कर्णोऽपरो भवेत् ।
यत्कृत्योर्वियुतिः सैकायुतिश्चैकोनिता कृतिः ।।

कोटिदोर्विवरादिष्टकर्णयोरन्तरं यथा ।
नाधिकं स्यात् तथा प्राज्ञ इष्टमत्र प्रकल्पयेत् ।।

iṣṭayorāhatirdvighnītyādyācāryokta mārgataḥ ।
koṭidoḥ śrutayaḥ sādhyāstatrakoṭibhujāhatiḥ ।।

dvinighnīparasaṁjñā syādiṣṭavargaparākhyayoḥ ।
yogāt tadantareṇāpitameko rāśirbhavettathā ।।

tenāntareṇa hṛdvighneṣṭaghnaḥ karṇo 'paro bhavet ।
yatkṛtyorviyutiḥ saikāyutiścaikonitā kṛtiḥ ।।

koṭidorvivarādiṣṭakarṇayorantaraṁ yathā ।
nādhikaṁ syāt tathā prājña iṣṭamatra prakalpayet ।।

Procedure:

Stage 1: With two *iṣṭa*s get the *karṇa* and right triangle, *para* = *koṭi* × *bhuja* × 2.

Stage 2: Now take one new *iṣṭa* where (कोटिदः विवरात् इष्टकर्णयोरन्तर (*koṭidaḥ vivarāt iṣṭakarṇayorantaraṁ*) (difference between इष्ट (*iṣṭa*) and कर्ण (*karṇa*) यथा न अधिकं स्यात् (*yathā na adhikaṁ sayat*)।

R_1 = the first *rāśi* = *iṣṭavarga* + *para* / *iṣṭavarga* − *para*.

R_2 = the second *rāśi* = 2 × *iṣṭa* × *karṇa*/*iṣṭavarga* – *para*.

Now R_1 and R_2 satisfy the condition $x^2 + y^2 - 1$ and $x^2 - y^2 + 1$ are both perfect squares.

Example Problem (1) (*Līlāvatī* (BDS), p. 24):

1, 2 taken as two *iṣṭa* sides of triangle are $x^2 - y^2$, $2xy$, i.e. 3, 4

Hyp^2 is $3^2 + 4^2$ is 5^2.

Karṇa = 5.

Para = *koṭi* × *bhuja* × 2 = 4 × 3 × 2 = 24.

We will start from 4.

Take *iṣṭa* = 4, *varga* = 16, add *para* = 24 + 16 = 40.

The difference between *iṣṭavarga* and *para* = 8.

The first *rāśi* = 40/8 = 5.

The second *rāśi* =द्विघ्नः कर्णः इष्टः/अन्तरम् (*dvighnaḥ karṇaḥ iṣṭaḥ/antaram*) = 2 × 5 × 4/5 = 40/8 = 5.

Now the two *rāśi*s are 5, 5 with $x^2 + y^2 - 1$ and $x^2 - y^2 + 1$ as 49, 1 whose square roots are 7, 1.

Similarly, when the *iṣṭa* = 5, the two *rāśi*s 49, 50 with $x^2 + y^2 - 1$ and $x^2 - y^2 + 1$ as 4900, 100 whose square roots are 70, 10.

Example Problem (2) (Līlāvatī (BDS), p .24):

2, 3 taken as two *iṣṭa* sides of triangle are $x^2 - y^2$, $2xy$, i.e. 5, 12

Hyp^2 is $x^2 + y^2$ is 169.

Karṇa = 13.

Para = *koṭi* × *bhuja* × 2 = 12 × 5 × 2 = 120.

We will start from 6.

Take *iṣṭa* = 6, *varga* = 36, add *para* = 36 + 120 = 156.

Difference between *iṣṭavarga* and *para* = 84.

The first *rāśi* = 156/84 = 13/7.

The second *rāśi* = द्विघ्नः कर्णः इष्टः/अन्तरम् (*dvighnaḥ karṇaḥ iṣṭaḥ/antaram*) = 2 × 13 × 6/84 = 13/7.

Now the two *rāśi*s are 13/7, 13/7 with $(x^2 + y^2) - 1$ and $(x^2 - y^2) +$

1 as 289/49, 1. These are squares as 289/49, 1 whose square roots are 17/9, 1.

Similarly, it can be proved when *iṣṭa* = 7, 8, 9 and 10.

There is yet another *sūtra* by Bapu Deva Shastri (*Līlāvatī* (BDS), p. 24):

Two *rāśi*s R_1 and R_2 are first squared. We find conditions on R_1, R_2 such that $R_2^2 + R_1^2 - 1$ and $R_2^2 - R_1^2 + 1$ are perfect squares.

इष्टस्य वर्गवर्गः सैकश्चेष्टाहतः प्रथमाराशिः।
इष्टकृतिकृतिर्द्विघ्नी रूपवियुक्ता भवेदपरः ।।

अनयोर्वर्गवियोगः सैको वर्गैक्यमेकहीनं च ।
वर्गः स्यादिष्टवशादेवं स्युरभिचराशयो बहुधा ।।

iṣṭasya vargavargaḥ saikaśceṣṭāhataḥ prathamārāśiḥ ।
iṣṭakr̥tikr̥tirdvighnī rūpaviyuktā bhavedaparaḥ ।।

anayorvargaviyogaḥ saiko vargaikyamekahīnaṁ ca ।
vargaḥ syādiṣṭavaśādevaṁ syurabhicarāśayo bahudhā ।।

Procedure:

The iṣṭa = x.

R_1 = the first *rāśi* = $(x^4 + 1)$ x; R_2 = the second *rāśi* = $(2x^4 - 1)$.

Now R_1 and R_2 satisfy the condition $x^2 + y^2 - 1$ and $x^2 - y^2 + 1$ are both perfect squares.

Example

The *iṣṭa* = 2; R_1 = the first *rāśi* = $(2^4 + 1)\ 2 = 34$; R_2 = the second *rāśi* = $(2 \times 2^4 - 1) = 31$.

$34^2 + 31^2 - 1$ and $34^2 - 31^2 + 1$ are both perfect squares. Square roots are 46, 14.

iṣṭa =3; R_1 = the first *rāśi* = $(3^4 + 1)\ 3 = 246$; R_2 = the second *rāśis* = $(2 \times 3^4 - 1) = 161$.

$246^2 + 161^2 - 1$ and $246^2 - 161^2 + 1$ are both perfect squares. Square roots are 294, 186.

Conclusion

Bapu Deva Shastri, besides being the Professor of Astronomy in Benares Sanskrit College, held many honorary posts such as the member of the Royal Asiatic Society of Great Britain and Ireland, member of the Asiatic Society of Bengal and fellow of University of Calcutta. He has made certain value additions to the topics of division, square, supposition, pulverization, progression, etc. for the benefit of better understanding of the students. This paper throws light on some of his techniques and examples as detailed in the book mentioned above. Thus, his contributions to ancient mathematics are praiseworthy.

References

Līlāvatī: *A Treatise on Arithmetic by Bhāskarācārya,* ed. Pandit Bapu Deva Shastri, Benares, 1883.

Līlāvatī, tr. H.T. Colebrooke, with Notes by H.C. Banerji, 2nd edn, New Delhi: Asian Educational Services, 1993.

Līlāvatī of Bhāskarācārya: *A Treatise of Mathematics of Vedic Tradition*, English tr. Krishnaji Shankhara Patwardhan, Somashekara Amrita Naimpally and Shyam Lal Singh, Delhi: Motilal Banarsidass, 2001.

17

Parikarmacatuṣṭaya and Pañcaviṁśatikā
A Study

V.M. Umamahesh

Introduction

THE HISTORY OF PARIKARMA THROUGH THE AGES

> बहुभिर्विप्रलापैः किं त्रैलोक्ये सचराचरे।
> यत्किञ्चिद्वस्तुतत्सर्वं गणितेन विना नहि।।

> Whatever there is in all the three worlds, which are possessed of moving and non-moving beings – all that indeed cannot exist as apart from *gaṇita.* What is the saying of good in vain?
>
> – Rangacharya 1912: 3

THE ancient Indian society was familiar with the all-pervasiveness of *gaṇita* that can be traced to Vedic period. Śulbasūtras give rules for constructing *vedī*s (sacrificial altars) and moves on to surds, etc.

Arithmetic and algebra are the two major fields in Indian mathematics. In arithmetic, there are various operations and out of which eight operations have been identified as fundamental. They are addition, subtraction, multiplication, division, square, square-root, cube and cube-root. A brief history of these fundamental operations (*parikarma*s) are presented here from the ancient up to medieval times.

Bhāskara I in his commentary on the *Āryabhaṭīya* states:

> All arithmetical operations resolve into two categories though usually considered to be four.
>
> अथ आचार्यार्यभटमुखारविन्दविनिसृतंपदार्थत्रयंगणितं, कालक्रिया, गोल इति यदेतद्गणितं तद् द्विधं चतुर्षु सन्निवष्टम। वृद्धिर्ह्रपचयश्चेति द्विविधम्। वृद्धिः संयोग, अपचयोह्लासः। एताभ्यां भेदाभ्यामशेषगणितं व्याप्तम्।
> संयोगभेदा गुणनागतानि शुद्धेश्च भागो गतमूलमुक्तम् ।
> व्याप्त समीक्ष्योपचयक्षयाभ्यां विद्यादिदं द्वयामकमेव शास्त्रम्।।
>
> That all mathematical operations are variations of the two fundamental operations of addition and subtraction was recognized by the Indian mathematicians from early times. The two main categories are increase and decrease. Addition is increase and subtraction is decrease. These two varieties of operations permeate the whole of mathematics (*gaṇita*). So, previous teachers have said: "Multiplication and evolution are particular kinds of addition; and division and involution of subtraction. Indeed, every mathematical operation will be recognized to consist of increase and decrease." Hence the whole of this science should be known as consisting truly of these two only. – Datta and Singh 1962: 130

PĀṬĪGAṆITA

Arithmetic is referred as *pāṭīgaṇita, dhūli-karma* or *vyakta gaṇita*. Algebra is referred as *bījagaṇita* or *avyakta gaṇita*.

> The word *pāṭīgaṇita* is a compound formed from the words *pāṭī,* meaning "board" and *gaṇita* meaning "science of calculation"; hence it means the science of calculation which requires the use of writing material (the board). The carrying out of mathematical calculations was sometimes called *dhūli-karma* (dust work) because the figures were written on dust spread on board or on the ground. Some later writers have used the term *vyakta gaṇita* (the science of calculation of the "known") for *pāṭīgaṇita*

to distinguish it from Algebra which was called *avyakta gaṇita* (the science of calculation of the "unknown").

– Ibid.: 123

Pāṭīgaṇita Works

Initially mathematics was included as a section in the astronomical works called Siddhāntas. Āryabhaṭa I (499) started this tradition. Later it became a general norm to include a section on mathematics in the Siddhānta works. Also, this developed into a separate stream over a period. The authors and their works which deal exclusively with *pāṭīgaṇita* are given below:

Author not known	– *Bakṣālī* manuscript (seventh century)
Śrīdharācārya	– *Triśatikā* (eighth century)
Mahāvīra	– *Gaṇitasārasaṁgraha* (ninth century)
Śrīpati	– *Gaṇitatilaka* (eleventh century)
Bhāskara II	– *Līlāvatī* (eleventh century)
Nārāyaṇa Paṇḍita	– *Gaṇitakaumudī* (fourteenth century)
Munīśvara	– *Pāṭīsāra* (seventeenth century)

In addition to the above popular works, there are many lesser known texts which deal exclusively with *pāṭīgaṇita*. In this paper, two fourteenth-century texts which deal exclusively with arithmetic operations are presented. They are the *Parikarmacatuṣṭaya* and *Pañcaviṁśatikā* both were edited and published by Takao Hayashi.

i. The *Parikarmacatuṣṭaya*, an anonymous Sanskrit work consists of versified rules and examples for the four fundamental arithmetical operations – *saṅkalita*, *vyavakalita*, *pratyutpanna* and *bhāgahāra*. Rules seem to be influenced by the *Triśatikā* but the examples are original. The addition and subtraction refer to sum of finite series of natural numbers and the difference between two finite series as can be found in the *Triśatikā*. This work contains 58 *śloka*s along with prose parts. All the examples quoted have been provided with answers.

According to the colophon of the manuscript, it was copied down for teaching the children of a Moḍha Baniā family. From the examples provided and the topics covered, it can be inferred that the main objective would have been to cover the topics useful for merchants (or would be merchants) for their day-to-day commercial transactions.

ii. Hayashi has edited and translated an another arithmetical work called the *Pañcaviṁśatikā*, based on two manuscripts (one from LD Institute, Ahmedabad and the other one from Oriental Institute, Baroda). Both the manuscripts contain Gujarati commentaries. As the name suggests, the original work should have had 25 *śloka*s. However, both the manuscripts contain more than that.

    The *Pañcaviṁśatikā* covers the topics of addition, subtraction, multiplication, division and other topics such as square root, rule of three, areas of square, investments, areas of triangle, area of circle, etc.

Addition

ADDITION IN PAÑCAVIṀŚATIKĀ

In this work, rule for addition and subtraction has been given:

यता स्थानकम् अङ्कानां युतिर्वियुतिरादितः।
खेनोनाद्यः स एव स न पाते।।

In this text the term used for addition is *yuti* and for subtraction is *viyuti*.

Beginning with the first numeral the sum or difference of numerals is made according to the places. That numeral is increased or decreased by zero itself.

Sum of the Series in Pañcaviṁśatikā

In this work, addition, sum of the natural series and arithmetic progression are all covered. Sum of the series of natural numbers is stated as:

सैकपदाहतपददलं एकादिचयेन भवति संकलितं।

द्विगुणिकृतसंकलितान् मूलं गच्छोवशिष्टसमः।।

Half of the product of the first (value) increased by unity and the number of terms will be that [sum which is obtained] by increasing one by one. Half of the sum of the square of the number of terms and first [value is also the sum]. Multiplying half of the first [or the first] increased by unity [by other value, one obtains] the [same] result.

$$S_n = 1 + 2 + 3 + \dots + n$$
$$S_n = \frac{n \times (n+1)}{2} \text{ or } \frac{n \times (n+1)}{2}.$$

The author provides another formula wherein:

व्येकपदघ्नचयः सद्योन्त्यं स्ववक्त्रयुतातद्दलम्।
मध्यं स्वपदनिघ्नं तत् सर्वस्वं जायते चये।।

a = first term (*ādhya*), d = common difference (*caya*), n = number of terms (*pada*) of an arithmetic progression. Last term is *antya* (a_n), middle term is *madhya* (m) and sum of the series is *sarvasva* (S_n).

$$a_n = a + (n-1) \times d,$$
$$m = \frac{(a + a_n)}{2},$$
$$S_n = nm.$$

ADDITION IN PARIKARMACATUṢṬAYA

The anonymous author begins this work with stating that "pair of procedural rules for addition, the first fundamental operation, is as follows":

आदौ रूपं ह्येकं रूपं चैकोत्तरं शतं गच्छः।
अन्तर्दशाद्दृष्टफलं वदन्ति सङ्कलित माचार्याः।।

First place is unity. Increase is also one up to one hundred. *Saṅkalita* is as per the revered *ācārya*, fruits seen at the interval of 10.

But it is not an addition but the sum of the first n terms of the natural series.

$S(n) = 1 + 2 + 3 \dots + n.$

That is,

Saṅkalita is $S(n) = 1 + 2 + 3 + ... + n$, where $n = 10k$ and $k = 1, 2, 3 \ldots 10$

The sum of a series starting with one and the common difference equal to one can be found out. But he states the method is to find out for the series $S(10)$, $S(20)$, …, $S(100)$.

यस्येच्छेत्सङ्कलितं राशिं तं तद्गुणं प्रकृत्यादौ।
प्रक्षिप्तम् हि तत्र हि च्छित्वार्धेन हि तत्फलमं भवति।।

> When one has multiplied the number whose *saṅkalita* one wishes to obtain by that number <same number> and added the <product> to the former, by half of the sum half <of the sum> the fruit *saṅkalita* shall be. – Hayashi 2007: 43

$$S_n = \frac{n^2 + n}{2}.$$

Vyavakalita: Subtraction

The terms used for subtraction by various authors are *vyutkalita, vyutkalana, śodhana, viyojana, viśodhana* and *viyoga*, and *āvaśeṣa, śeṣa* and *āvaśeṣaka* are the terms used for remainder.

A few authors define normal subtraction stating that according to their places (units, tens, etc.) difference is to be found out.

SUBTRACTION IN PAÑCAVIṀŚATIKĀ

In this work, like in the *Triśatikā*, *vyavakalita* refers to the difference between the sums of two natural series (v. 3):

संकलितोत्पन्नद्युम्नाद् व्ययं त्यक्त्वा धनं भवेत्।
तद्धनं व्यवकलितं मुनिभिः पुरा।।

> Having subtracted the expense (*vyaya*) from the property (*dyumna*) produced by addition (*saṅkalita*), there will be property (*dhana*). This property has been called the difference (*vyavakalita*) by the ancient sages. – Hayashi 1991: 415

$Sn - n = S_{n-1}$

Ex. $n = 10$. $S_{10} - 10 = 55 - 10 = 45 = S_9$.

Saṅkalita (addition) is an elementary function. Hence it was not dealt in detail by various astronomical works. But *saṅkalita* is also referred to summation. The sum of natural numbers and the sum of series are described by various texts.

SUBTRACTION IN PARIKARMACATUṢṬAYA

In this work, a different "subtraction" is explained (vv. 11-13). the sum of a natural series up to a chosen number is deducted from the sum of the natural series from 1 to 100 and is defined as *saṅkalita* of that chosen number.

सङ्कलितै सङ्क्षेपो व्यवकलिते यद्वत्तद्वत् क्षयोपि कर्त्तव्यः।
अन्तर्दश दृष्टफलम् वदन्ति सङ्कलितमाचार्यः।।

यस्येच्छेद्व्यवकलितम् तस्मिन्नेकोत्तर शतम् दद्यात्।
तत्र हि शतगुणम् च दलीकृतम् व्यवकलितमाहुः।।

शतवर्गात् शतमिश्राद् दलिताद् व्यवकलितराशिमादिष्टम्।
व्यपहृत्य ततः शेषाः पूर्वविधानेन गच्छः स्यात्।।

Addition (*sankṣepa*) is made in *saṅkalita*; subtraction (*kṣaya*), too, should be made in *vyavakalita*. The revered professor calls the fruits seen at the interval of ten *vyavakalita*.

One should add one hundred and one to that (number) whose *vyavakalita* one wishes to obtain. (The sum is) multiplied by one hundred decreased by that (number) and halved; they call it *vyavakalita*.

From the square of one hundred increased by one hundred and halved, the specified value of *vyavakalita* is subtracted. From that remainder, by means of the previous rule, the step (the number of terms) shall be (obtained). – Hayashi 2007: 46

Let V_n be the *vyavakalita* of n. Its definition,

$V_n = S(100) - S_n$,

$V(10) = S(100) - S(10)$

$V(10) = 5050 - 55 = 4995$.

Multiplication

Out of four fundamental arithmetical operations, multiplication has been dealt in detail and various methods of operation have been provided by our sages. To summarize, the methods are:

i. *Kapāṭasandhi*
ii. *Gomūtrikā*
iii. *Khaṇḍa*
   a. *Rūpa-vibhāga*
   b. *Sthāna-vibhāga*
iv. *Bheda*
v. *Iṣṭa*
vi. *Tatastha*
vii. Special method appearing in the *Gaṇitamañjarī* (Gelosia or Grating method).

The modern method of multiplication has already been in practice here. The evolution of the methods is in line with the progress in the writing materials. Earlier methods act as building blocks on which new methods are invented.

MULTIPLICATION IN PAÑCAVIṀŚATIKĀ

The *Pañcaviṁśatikā* enumerates four methods of multiplication. They are *kapāṭasandhi, gomūtrikā, tatastha* and *khaṇḍa*. As per the text, *kapāṭasandhi*, *gomūtrikā* and *tatastha* each is of two kinds and *khaṇḍa* is of three kinds (v. 4).

> द्विधा कपाटसन्धिश्च तथा गोमूत्रिका द्विधा।
> तस्थो द्विधा पुनः प्रोक्तस्तथा खण्डा त्रिधा स्मृतः।।

Hayashi opines that:

> Due to laconic expressions of the versified rules and sketchy descriptions of the commentaries, there remains much ambiguity about the details of the procedures. – Hayashi 1991: 417

MULTIPLICATION IN PARIKARMACATUṢṬAYA

This text lists four methods for multiplication (vv. 20-21):

> राशिं विन्यस्योपरि कपाटसन्धिक्रमेण गुणराशेः।
> अनुलोमविलोमाभ्याम् मार्ग्राभ्याम् ताडयेत्क्रमशः।।

तस्थः प्रत्युत्पन्नः खण्डो द्विविधः कपाटसन्धिश्च ।
करणचतुष्टयमेतत् प्रत्युत्पन्ने विनिर्दिष्टम् ।।

Having put down the number <to be multiplied> above the multiplier in the manner of "door junction" (*kapāṭa-sandhi*), one should multiply <the digits> one by one in regular or reverse order.

The multiplication called "standing there" (*tatastha*), two kinds of parts (*khaṇḍa*) and "door-junction": these are the quartet methods told for multiplication (vv. 20-21). – Hayashi 2007: 48

The author has followed the four methods told by Śrīdharācārya.

Śrīdharācārya describes *kapāṭasandhi* as (vv. 5-6ab):

विन्यस्याधो गुण्य कपाटसन्धिक्रमेण गुणराशेः।
गुणयेत् विलोमगत्याऽनुलोममार्गेण वा क्रमशः।।

उत्सार्यौत्सार्य ततः कपाटसन्धिर्भवेदिदं करणम्।

Having placed the multiplicand (*guṇya*) below the multiplier (*guṇa-rāśi*) as in the junction of two doors, multiply successively in the inverse or direct order, moving (the multiplier) each time. This process is known as *kapāṭasandhi*.

Tatastha means being there or stationery. Śrīdharācārya explains this in his *Triśatikā* (v. 6cd):

तस्मिंस्तिष्ठति यस्मात् प्रत्युत्पन्नस्ततस्तत्स्थः।।

When the *pratyutpanna* is performed by keeping the multiplier stationary, the process is called *tatastha* (multiplication) at the same place.

This is of two varities, according to Śrīdhara (*Triśatikā* v. 7 and *Pāṭīgaṇita* v. 20).

रूपस्थानविभागात् द्विधा भवेत्खण्डसंज्ञकं करणम्।
प्रत्युत्पन्नविधाने करणान्येतानि चत्वारि।।

The process of multiplication is called *khaṇḍa* (or *khaṇḍa-guṇana*, "multiplication by parts") is of two varieties (called *rūpa-vibhāga*

> and *sthāna-vibhāga*), depending on whether the multiplicand or multiplier is broken up into two or more parts whose sum or product is equal to it, or the digits standing in the different notational places (*sthāna*) of the multiplicand or multiplier are taken separately. – Shukla 1959: 13-14

Division (Bhāgahāra)

David Eugene Smith, in his *History of Mathematics* (1953) states:

> The operation of division was one of the most difficult in the ancient *logistica*, and even in the fifteenth century it was commonly looked upon in the commercial training of the Italian boy as a hard matter. Pacioli (1494) remarked that "if a man can divide well, everything else is easy, for all the rest is involved therein". – Vol. 2: 132

The process of division was considered to be too tedious by the European scholars even during fifteenth century, whereas *siddhānta* authors (fifth century) considered division as too elementary to be described.

In almost all *siddhānta* works, methods of division are not explained. But division is used in other calculations. But in *pāṭī* works we can find that division methods are explained with examples. It is evident from those works that our sages knew the modern method of division then.

The common Indian names for division are *bhāgahāra*, *bhājana*, *haraṇa*, *chedana*, etc. All these terms literally mean "to break into parts", i.e. "to divide", excepting *haraṇa* which denotes "to take away". This term shows the relation of division to subtraction. The dividend is termed *bhājya*, *hārya*, etc.; the divisor *bhājaka*, *bhāgahāra* or simply *hara*, and the quotient *labdhi* "what is obtained" or *labdha* (Datta and Singh 1962: 131).

DIVISION IN PAÑCAVIṀŚATIKĀ

Having put down the divisor below the question (the dividend) and divided the question by the divisor, the division should be made (part should be taken away) in order. (Thus) the rule division has been certainly handed down (to us).

प्रश्नाद् अधो हरं न्यस्य प्रश्नं छित्वा हरेण च।
भागो हार्यः क्रमान् नूनं भागाहारविधिः स्मृतः।।

This gives the well-known method which places the divisor (*hara*) and the divided, which (the latter) is called "the (number in) question" (*praśna*) in our text.

Example

1	6	2	0
1	2		

1		
4	2	0
1	2	

1	3	
	6	0
	1	2

1	3	5
		0
	1	2

DIVISON IN PARIKARMACATUṢṬAYA

The *Parikarmacatuṣṭaya* explains the method of division. The text states (vv. 37-38):

राशि विन्यस्याधोराशिनिहारको विशोध्यस्तु।
उपरिमराशेः क्रमशः प्रतिलोमम् भागहारपुञ्जेन।।

भाज्यम् हारम् च द्वौ तुल्येन हि राशिना सशिन्य सदा च्छित्वा।
शेषम् च्छेदविभक्तम् फलमथ भागात्मकम् भवति।।

When one has put down two numbers (one above the other), the lower number, which is the divisor, should be subtracted (from the upper number) one by one in reverse order. This is a rule for division.

One should always divide the two, the dividend and the divisor, by the same number; the quotient (lit. the remainder) (from the dividend) divided by (the quotient from) the divisor is a fruit (quotient) that has the nature of division. – Hayashi 2007: 52

The verses are based on Śrīdhara's *Triśatikā* (v. 9).

तुलेयन संभवे सति हरं विभाज्यं च राशिना छित्त्वा।
भाहे हार्यः क्रमशः प्रतिलोमं भागहारविधिः।।

An example from the *Parikarmacatuṣṭaya* for division (v. 42):

अयुतत्रयम् सहस्रमेकम् सद्विशतम् षट्सप्ततिसमायुक्तम्।
सप्ताशीत्या भक्तम् प्रकथयमेकभागाख्यम्।।

न्यासः 30276/87 = 348

> Three *ayuta*s, two hundred and seventy-six <gold pieces> were divided by eighty-seven <men>. What is the share for one should be told.

Dividing 3027 by 87 gives the answer of 348. Each will get 348 gold pieces.

Prime Numbers

An important observation made by Hayashi in this text is the occurrence of nine large prime numbers greater than 100 which he thinks cannot be a coincidence.

> This high frequency indicates that it cannot be a coincidence. The author of the present form of that part, at least, must have intentionally used two primes to construct his examples for division. – Hayashi 2007: 21

Hayashi calls primer number as *accheda* which has no divisor. This is the first-time prime numbers surface in ancient Indian mathematical work. It will definitely be a subject matter for a separate research.

Today, prime numbers are used in cryptography for network security. Cryptography or cryptology is the practice and study of techniques for secure communication in the presence of third parties called adversaries.[1]

Conclusion

Hayashi (1991: 404) states:

> From the viewpoint of the history of Indian mathematics, the importance of our text lies in its historical expansion and reformation rather than in its mathematical contents, as it throws new light upon history of reformation of other Sanskrit mathematical treatises.

It applies to both the anonymous works discussed in this paper.

The title of the work the *Pañcaviṁśatikā* reminds Śrīdharācārya's

[1] https://en.wikipedia.org/wiki/Cryptography

Gaṇitapañcaviṁśati and Tejasiṁha's *Iṣṭāṅgapañcaviṁśatika*. Hayashi (1991: p. 405) points out:

> These two works, devoted for particular topics, may be regarded as a kind of monograph, which is hitherto a neglected field of study in Indian mathematical literature.

Introducing the students to the simple and clear *Pañcaviṁśatikā* like texts will allay their fears about the complexity of the ancient works and will attract more students to a serious study of the ancient texts. More such studies based on modern sciences will bring to light the marvellous discoveries of our scholars of those times.

References

Datta, B.B. and A.N. Singh, 1962, *History of Hindu Mathematics,* 2 vols, Bombay: Asia Publishing House.

Hayashi, Takao, 1991, "Pañcaviṁśatikā: In Its Two Recensions", *Indian Journal of History of Science*, 26(4), p. 395-448.

———, 2007, "*A Sanskrit Arithmetical work in a fourteenth century Manuscript*", *The Journal of Oriental Research*, vols. 74-77, p. 19-58.

Rangacharya, M., 1912, *Gaṇitasārasaṁgraha of Mahāvīra*, Madras: Government Press.

Shukla, K.S. 1959, "*Paṭīgaṇita of Śridharācārya*", Department of Methematics and Astronomy, Hindu Mathemetical and Astronomical Text Series no. 2, Lucknow: Univerisity of Lucknow.

Smith, David Eugene, 1925, "*History of Mathematics,*" vol. 2, New York: Dover Publications Inc., originally published in 1923.

18

An Appraisal of Vākyakaraṇa of Parameśvara

Venketeswara Pai R.

Abstract: Vaṭaśśeri Parameśvaran Nambūdiri popularly known as Parameśvara (1380–1460) was a mathematical astronomer of the Kerala school of astronomy and mathematics founded by Mādhava of Saṅgamagrāma. He has authored several works including Dṛggaṇita which is composed by revising the parameters based on observations. The text *Vākyakaraṇa* of Parameśvara is unique in the sense that it gives algorithm for constructing the *vākya*s. It is mentioned in the second half of the first verse of text that:

करोति वाक्यकरणं वाक्यावयवसिद्धये

The text *Vākyakaraṇa* is composed for obtaining the *vākya*s.

The *Vākyakaraṇa* contains sixty-six verses and gives algorithm for obtaining the *vākya*s such as *gīrṇaśreyādi-vākya*s, *saṅkrānti-vākya*s and so on. In this paper having given an overview of the text, we would proceed to explain some of the algorithm for obtaining the *vākya*s. We have used the paper manuscript (MS KVS 242) for our study. This manuscript was collected from K.V. Sarma Research Foundation where it is preserved. Sarma transcribed this from the manuscript (MS Triv. C. 133A.) which is preserved in Travancore University Manuscripts Library, Trivandrum. In this article, we shall have a brief overview of the text.

Keywords: *Vākyakaraṇa*, *vākya*s, Kerala school, Parameśavara.

Introduction

The Kerala school of Indian astronomy and mathematics, that flourished for more than four centuries starting from Mādhava (1350 CE) of Saṅgamagrāma, is well known for its contributions to mathematics, in particular to the branch that goes by the name of mathematical analysis today. Besides making several important contributions to mathematical analysis which includes discovering the infinite series for sine, cosine and arc tangent functions, as well as its fast convergent approximations, the astronomers of the Kerala school have also made significant contributions to the advancement in astronomy, particularly the planetary theory.

Pioneered by Mādhava (*c*.1340–1420) and followed by illustrious mathematicians and astronomers like Parameśvara, Dāmodara, Acyuta and others, the Kerala school extended well into the nineteenth century as exemplified in the work of Śaṅkaravarman (*c*.1830). Only a couple of astronomical works seem to be extant now. Most of Mādhava's celebrated mathematical discoveries – such as the infinite series for "pi", its fast convergent approximations and so on – are available only in the form of citations in later works. Mādhava's disciple Parameśvara (*c*.1380–1460) is reputed to have carried out detailed observations for over fifty years and composed a large number of original works and commentaries. Among his works, the *Dṛggaṇita* finds its position at the first place. The *Vākyakaraṇa* is another work of Parameśavara in Vākya school of astronomy.

Vākya School of Astronomy

The huge corpus of astronomical literature that has been produced in India from the time of Āryabhaṭa (*c*.499 CE) is generally divided into Siddhāntas, Tantras, Karaṇas and Vākyas; in decreasing order of the theoretical contents astronomical parameters given in Siddhāntic texts are very large. In these texts, complex and lengthy computational algorithms are employed in finding the planetary longitudes and other astronomical quantities. Hence, evolved a new school of astronomy which is known as the Karaṇa school. The epoch is chosen to a closer date and observed planetary longitudes documented. Astronomical parameters are made smaller in magnitude. The Karaṇa texts describe the simplfied algorithms and the mathematical equations are modified for computational ease.

The *vākya* method of finding the true longitude of the sun, the moon and the planets (*sphuṭagraha*) is a brilliantly designed simplified version of the methods outlined in the various Siddhāntas. As per the Siddhāntas, we first find the mean longitudes of the planets and then apply a few *saṁskāra*s. The *manda-saṁskāra* is to be applied in the case of the sun and the moon, whereas both the *manda-saṁskāra* and *śīghra-saṁskāra* are to be applied in the case of the other five planets to get their true positions. On the other hand, the *vākya* method, by making use of a few series of *vākya*s presents a shortcut directly leading to the true longitudes of the planets at certain regular intervals, starting from a certain instant in the past. We will discuss about this instant, which is also closely linked with other notions such as *khaṇḍa* and *dhruva*, during the course of our discussion. At this stage it would suffice to mention that this *vākya* method provides a simple elegant method for computing the true longitudes without having to resort to the normal procedure of calculating a whole sequence of corrections involving sine functions, etc. which would be quite tedious and time consuming. Therefore, the *vākya* method became very popular in south India and even today some *pañcāṅga*s are brought out using the *vākya* method in the southern states of India (Pai et al. 2018).

TEXTS RELATED TO VĀKYA *S*YSTEM OF ASTRONOMY

The earliest literature on *vākya*s can be traced back to the time of Vararūci and it is known as *gīrṇaḥ-śreyādi-vākya*s. It is the set of 248 *vākya*s which gives the true longitudes of the moon for 248 consecutive days. Hence, it is also known as *candra-vākya*s. Since, these *vākya*s have composed by Vararūci, it is popular by the name Vararūci-*vākya*s. These give the longitude of the moon correct up to the minutes. Mādhava gives another set of *candra-vākya*s which is known by the name Mādhava-*vākya*s. These are accurate up to the seconds. The canonical text of the Parahita system, the *Grahacāranibandha* of Haridatta (seventh century), introduces *vākya*s for the *manda* and *śīghra* corrections which are referred to as the *manda-jyā*s and *śīghra-jyā*s.

The fully developed *vākya* system is presented in the famous *karaṇa* text of the thirteenth century, the *Vākyakaraṇa*, which gives the method of directly computing the true longitudes of the sun, the moon and the planets using *vākya*s. Manuscripts of this work are available in various manuscript libraries of south India, especially of Tamil Nadu. Kuppanna Sastri and K.V. Sarma estimate that it was composed between 1282 and 1306 CE. The author of this work is not known, but probably hailed from the Tamil-speaking region of south India. It has a commentary called the *Laghuprakāśikā* by Sundararāja who hailed from Kāñcī near Chennai. The work is based on the *Mahābhāskarīya* and the *Laghubhāskarīya* of Bhāskara I belonging to the Āryabhaṭa school, and the Parahita system of Haridatta prevalent in Kerala.

The *Vākyakaraṇa* and the other works pertaining to the Vākya system only present the lists of *vākya*s and the computational procedures for obtaining the longitudes of the planets using these *vākya*s. However, the *Vākyakaraṇa* of Parameśavara gives the rationale behind some of the Vākyas. Thus, it is an important text in the *vākya* school of astronomy.

Vākyakaraṇa of Parameśvara

THE AUTHOR

Parameśvara was one of the reputed mathematician-astronomers of the Kerala school who seems to have flourished around the beginning of fourteenth century and was a pupil of Mādhava. Parameśvara proposed several corrections to the astronomical parameters which had been in use since the times of Āryabhaṭa based on his eclipse observations. The computational scheme based on the revised set of parameters has come to be known as the Dṛk system. The text composed based on the system is called the *Dṛgganita*. Parameśvara mentions in his work *Dṛggaṇita* that he has composed the same in the Śaka year 1353 (Sarma 1963).

Based on an old manuscript of a Malayalam commentary on the *Sūrya-Siddhānta* preserved in the Oriental Institute, Baroda, MS No. 9886, contains in the statements:

> *parameśvaran vaṭaśśeri nampūri, nilāyāḥ saumyātīrasthaḥ parameśvaraḥ ... asya tanayo dāmodaraḥ, asya śiṣyo nīlakaṇṭhasomayājī, ...*
>
> Parameśvara was a Nampūri from Vaṭaśśeri [family]. He resided on the northern bank of the Nīlā [River]. ... His son was Dāmodara. Nīlakaṇṭha Somayājī was his pupil. ...
>
> – Sarma and Hariharan 1991

From the first verse of the *Vākyakaraṇa*, it is evident that the author of the work is Parameśvara.

> पूज्यपादस्य रुद्रस्य शिष्योऽयम् परमेश्वरः ।
> करोति वाक्यकरणं वाक्यावयवसिद्धये ।।
>
> *pūjyapādasya rudrasya śiṣyo 'yam parameśvaraḥ* ।
> *karoti vākyakaraṇaṁ vākyāvayavasiddhaye* ।।
>
> Parameśvara is the student of the venerable Rudra. [The work] *Vākyakaraṇa* is done for obtaining the *vākya*s.

Here, the teacher "Rudra" is none other than the father of Parameśvara. Apart from his father, Mādhava was also the teacher of Parameśvara. The second line of the verse states the purpose of the text. That is, the rationale for the *vākya*s or it gives the procedure for obtaining the *vākya*s.

THE TEXT

The manuscript of the text *Vākyakaraṇa* (MS no. KVS 242) has 15 folios written in Malayalam script and the language is Sanskrit. The *Vākyakaraṇa* of Parameśvara is a small and an important treatise in *vākya* system. It contains sixty-seven verses in total. The beginning verses of the text provide the rationale for *gīrṇa-śreyādivākya*s. Later, a couple of verses emphasize the importance of the corrections such as *deśāntara* and *dhruva-saṁskāra*. for obtaining the true longitude of the moon. A brief content of the text is as follows:

- First verse states the authorship and pupose of the text.
- Next two and half verses give the rationale for obtaining the *gīrṇa-śreyādi-candra-vākya*s.
- After this, the author emphasizes on the importance of applying *deśāntara* and *aharmāna* corrections for obtaining the true longitude of the moon in one and a half verses.
- Seven verses (6-13) explain the procedure for applying the *aharmāna* corrections.
- Verses 14 to 25 describe the procedure for obtaining *dhruva-saṁskāra-hāraka.*
- Next five verses explain the rationale for obtaining the *yogyādi-vākya*s.

These are set of forty-eight *vākya*s used to compute the true longitude of the sun at any desired instant. The text *Karaṇapaddhati* of Putumana Somayājī gives the rationale for *yogyādi-vākya*s. For more details regarding the *yogyādi-vākya*s, see Pai et al. (2018) and Pai et al. (2015).

- Rationale for *saṅkrānti/saṅkramaṇa-vākya*s are explained through verses 32 to 40. The *saṅkrānti-vākya* is the time interval between the *meṣa-saṅkrānti*, and any *saṅkrānti*, expressed in a *vākya*.
- Later verses talk about the need of *dhruva-saṁskāra* and explain it in a different manner. While doing so, it also talks about the use of *candra-vākya*s more efficiently also that the error accumulated would be minimum.

It is to be noted that one of the important topics that is mentioned in the *Vākyakaraṇa* of Parameśvara is the *dhruva-saṁskāra-hāraka.* It is the divisor in a correction term which is known as *dhruva-saṁskāra.* As name suggests, *t*his is a correction term which is to be applied to the *dhruva* of the moon. This is applied in the context where the moon's true longitude is found using the *vākya* method. The detailed explanation of the procedure for obtaining the true longitude of the moon is found in the *Vākyakaraṇa* of thirteenth century (Pai et al. 2009; Pai et al. 2018; Sastri and Sarma 1962). The true longitude obtained here is slightly deviated from the actual value. This error arose because of the *dhruva* corresponding to the number of days of cycle of anomalistic revolutions. Significance of these anomalistic cycles is that the day on which the cycle is completed the moon's anomaly should be zero at the sunrise. In actual, there would be a small finite value for longitude of anomaly at the sunrise. The entire algorithm, for finding the true longitude, is based on the assumption that at the end of each anomalistic cycle, the anomaly would be zero at the sunrise. Hence, it is necessary to correct the obtained longitude in order to get the accurate value of the true longitude. However, the *Vākyakaraṇa* of thirteenth century does not talk about this correction. It is the *Vākyakaraṇa* of Parameśvara which gives a detailed explanation regarding this correction term which goes by the name *dhruva-saṁskāra.* The verses which describe the *dhruva-saṁskāra* and their translation are given below:

आनीय तुङ्गमध्येन्दू वाक्यारम्भदिनोदये ।
तयोरन्तरमानीय तेनेन्दोर्मध्यमां गीतम्।।

चन्द्रोच्चभुक्तिरहितां विभजेल्लब्धमत्र तु।
ध्रुवसंस्कारसंज्ञः स्याद्धारकः स्वर्णसंज्ञितः।।

Having obtained the mean longitudes of the moon and its anomaly at the sunrise on the day when the counting of the *vākya*s starts and having obtained their difference, and by that [difference] the difference in rates of motion of the moon and its apogee is to be divided. This is called as *dhruva-saṁskāra-hāraka.* This has both positive and negative nature.

The above verses give only the "denominator" part of the correction term. The whole correction term is to be applied negatively to the longitude of the moon when the longitude of the apogee is greater.

Otherwise, it has to be applied positively. The following verse explain the same.

तुङ्गेऽधिके ऋणाख्यः स्याद्धघनाख्यधिके विधौ ।
ध्रुवसंस्कारसंज्ञं तु हारकध्रुववत् पठेत् ॥

The above verse gives the condition when the correction is applied negatively or positively. The next verse explains the entire correction term.

"रत्नश्रेये" ति संशोध्य स्वोदयस्फुटभुक्ति तः ।
हारकेण विभज्याप्तं स्वर्णं कुर्यान्निशाकरे ॥

> The *ratnaśreya* (12°02') is to be subtracted from the true rate of motion of the Moon. [The result] has to be divided by the *hāraka*. [What is obtained here] has to be applied positively and negatively to the longitude of the moon.

The term *ratnaśreya* gives the numerical value of the rate of motion of the moon when it has the slowest motion. The value encoded in the term *ratnaśreya* is 12° 02'. This is the value when the moon has the slowest motion. This happens when the moon coincides with its apogee. In other words, it is the rate of motion of the moon when its anomaly is zero.

Concluding Remarks

From the study, it is clear that the *Vākyakaraṇa* of Parameśvara acts as an appendix to the *Vākyakaraṇa* of thirteenth century. In fact, it fills the gap by introducing the unexplained topics such as *dhruva-saṁskāra*. The purpose stated in the first verse:

करोति वाक्यकरणं वाक्यावयवसिद्धये

> The text *Vākyakaraṇa* is composed for obtaining the *vākyas*, and the *vākyas* has also been served by the text, as the text dedicates itself for giving the rationale for *vākya*s.

References

Pai, Venketeswara R., D.M. Joshi and K. Ramasubramanian, 2009, "The Vākya Method of Finding the Moon's Longitude", *Gaṇita Bhāratī*, 31(1-2): 39-64.

Pai, Venketeswara R., K. Ramasubramanian and M.S. Sriram, 2015, "Rationale for Vākyas Pertaining to the Sun in Karaṇapaddhati", *Indian Journal of History of Science*, 50(2): 245-58.

Pai, Venketeswara R., K. Ramasubramanian, M.S. Sriram and M.D. Srinivas, 2018, "Karaṇapaddhati of Putumana Somayājī: Translation with Detailed Mathematical Notes", *HBA* 2017; Springer.

Sarma, K.V., 1963, *Dṛgganita of Parameśavara*, critical edition, Hoshiarpur: Vishveshvara and Vedic Research Institute.

Sarma, K.V. and S. Hariharan, 1991, "Yuktibhāṣā of Jyeṣṭhadeva: A Book of Rationale in Indian Mathematics and Astronomy in Analytical Appraisal", *Indian Journal of History of Science*, 26: 185-207.

Sastri, T.S. Kuppanna and Sarma K.V., 1962, *Vākyakaraṇa with the Commentary by Sundararāja*, Madras: KSRI.

19

Astronomical Observations and the Introduction of New Technical Terms in the Medieval Period

B.S. Shylaja

Abstract: The science of astronomy developed from observations. These aspects are covered in almost every textbook by Indian astronomers. However, the finer details on the instruments, observational procedures, corrections and errors need to be studied systematically. Since the measurable quantities are only angles and time, the descriptions are generally brief. In an attempt to extract the observational procedures relevant parts of the texts are highlighted. This also throws light on some new unknown words perhaps coined for the need. This is especially true in the texts of late nineteenth century when the usage of telescopes was being introduced. A list of such new words will be presented and discussed.

Keywords: Observational astronomy, Indian texts, medieval period, new technical terms.

Introduction

THE observational aspects of Indian astronomers are covered in almost every textbook on astronomy. However, the finer details on the instruments, observational procedures, corrections and errors are not explicitly mentioned and therefore need to be extracted systematically. The measurable quantities are only angles and time; moreover, the descriptions are generally brief. For example, the introduction of subdivision for *aṅgula* as *vyaṅgula* is noticeable in many texts. However, the exact definition of the fraction and the method to measure are not indicated in any text. The fraction of a degree is written down in many texts and the method of measurement is not described. The accuracies achieved appear to indicate that they are calculated values. However, the basic parameters that are measured also are seen to be of accuracies of 1 arc minute. Here we will discuss the development of observations and procedures by broadly classifying them into three categories – the Siddhāntic period (up to about twelfth century), medieval period (up to about seventeenth century) and the colonial period giving typical examples of observations and the associated coinage of new technical terms.

Clues in Siddhāntic Texts

We look for clues about instruments used for observations in the Siddhāntic texts like the *Āryabhaṭīya* and the *Siddhānta Śiromaṇi*. It is interesting to see that the angles are measured in terms of time. The exhaustive work on these instruments (Ohashi 1994) has demonstrated the use of various instruments and the accuracies achieved. However, the role of later astronomers in improving the accuracies does not get highlighted. Here is one example.

It is well known that the declination of the sun changes during the year from 23.5° N to 23.5° S. Generally for all calculations this value is taken as the same for any given day. However, between the sunrise and the sunset there is a small change in the declination. This varies throughout the year. Ohashi (1997) has shown that a correction to this effect also was measured; and was incorporated in all calculations. This was called *apacchāyā*. This word does not find a place in many texts. It is mistaken with *avachāyā* (penumbra) by some. He discusses the various interpretations, such as "wrongly placed shadow" and "reduced shadow" which could not point to the need for the correction itself.

The declination of the sun is given by sin δ = sin λ sin ε, where δ is the declination, λ, the longitude and ε, the obliquity of the rotation axis of the earth.

The increase in λ is about a degree per day. Therefore, from sunrise to sunset the change is about ½ degree. For values near λ = 0 or 180, the difference of ½ degree would not give appreciable difference. However at solstices for an increase in λ by ½ degree can result in a change in the value of δ which is not negligible. This has been indicated in the texts *Mānasāra* and *Mayamata* (*fig*. 19.1). The word *apacchāyā* was interpreted as a correction for this and the logic remained unknown till Ohashi revealed it.

The interpretation of the new technical terms, therefore, demands understanding of the observational technique itself. This also takes us to the question of what were the observations that were carried out and how they were interpreted to derive parameters pertaining to the details of orbit.

For example, the parallax of the moon as defined in modern terminology is the angle subtended by the moon at the radius of the earth. This quantity is essential for all calculations pertaining to eclipses. One needs to measure the position of the moon very accurately every night. This can be achieved using versed sine

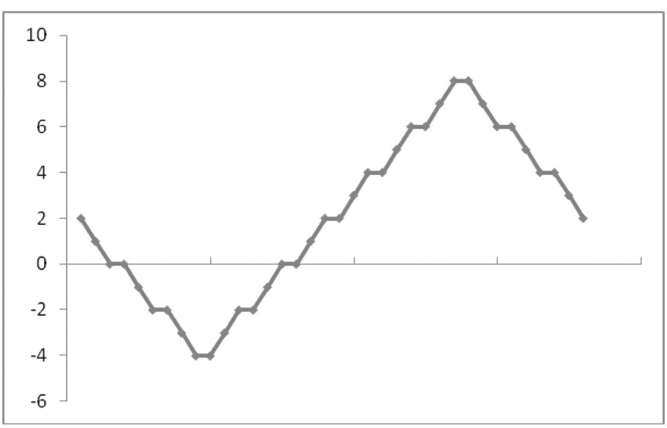


fig. 19.1: *Apacchāyā* corrections through the year as provided in the *Mānasāra* and *Mayamata*; this was interpreted as variation of the noon shadow expressed in a modified linear zigzag fashion ratio (*utkrama jyā*) as will be explained later. However, this is not mentioned in any text. Based on this measurement, the daily motion of the moon is given as 790'35''. The angular size of the moon is another quantity measured (16'4½'').

The quantities derived from these measurements are:

a. The moon's daily motion is given as fifteen times the radius of earth.

b. The moon needs 4 *ghaṭī*s to traverse this distance.

c. 11854¾ *yojana* is the distance covered.

d. The earth's radius is 1581/2 *yojana*s.

While the observations of the sun and the moon are carried out with a gnomon and a simple angle-measuring device, the technique for observing the planets is not explicitly discussed anywhere. Here is a hint on observations of planets in a verse in *Grahalāghava* 10.4 (*fig*. 19.2). It reads:

> The reflection of a planet is first seen. The *lamba* is measured from the (horizontal) ground level to the point of reflection. The distance between the foot of the *lamba* and the point of reflection is measured in *aṅgula*s. This is the *bhujā*. This value multiplied by 12 and divided by the elevation of the reflected point. The result gives the *chāyā* in *aṅgula*s. – Rao 2006

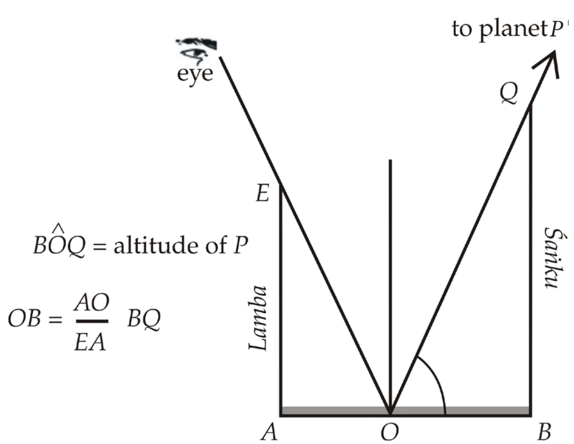


fig. 19.2: Method described in the *Grahalāghava* for getting the shadow length of planets

It is intended that the reflection was from the surface of water. A verification of this method for deriving the lunar eclipse timings was attempted recently and failed. However, if oil is used instead of water, meaningful measurements are obtained (K.G. Geetha, personal communication).

Astrolabes: Measures of Coordinates, Description of Measurements

Astrolabes were introduced in India around thirteenth century. The texts devoted to the construction and use of this instrument describe the measurement procedures quite in detail. Mahendra Sūri translated the manual of using the astrolabe in thirteenth century; this was followed by commentary by Malayendu. Subsequently many more texts followed – notably the *Siddhāntarāja* by Nityānanda (Sarma 2018). Here the conversion of time measure to angle is eliminated since the angles are measured directly. In this context, the procedures of using tabletop instruments also emerge. For example, the altitude and azimuth are measured for any object. They need to be converted to longitudes and latitudes which can be done as formulae. Further conversion of these into right ascension and declination are also done the same way. We see that the measured quantities like the *paramonnatāṁśa* (maximum altitude) are listed in minutes of arc. All the other quantities are calculated and hence are listed to arcseconds (Venkateswara and Shylaja 2016: 1551).

In this context it may be worth mentioning the uniqueness of the trigonometric ratio called *utkrama-jyā* (versed sine). As is well known, a counterpart of this does not exist in European texts. But its advantages and uses are well known. A new ratio have (half versed sine) was defined as

$$\text{hav}(\theta) = (1 - \cos\theta)/2 = \sin^2(\theta/2). \qquad (1)$$

It finds a unique application in navigational measuring devices. If one needs to measure distance between two locations on the earth based on the (measured) longitude and latitude, the procedure was simplified by approximating it to a plane triangle and by applying the Pythagoras Theorem. This implies working out square roots which was a tedious procedure 200 years ago. Here, by using the versed sine one could get solutions quickly.

$$\text{hav } s = \text{hav } \Delta\phi + \cos\phi 1 \cos\phi 2 \text{ hav } \Delta\lambda, \quad (2)$$

where s is the angular separation (which multiplied by radius of the earth is the minimum distance on the sea/land) between two stations with longitudes and latitudes as $\lambda 1$, $\phi 1$ and $\lambda 2$, $\phi 2$, $\Delta\phi$ is the difference in latitudes and $\Delta\lambda$ is the difference in longitudes.

The technique, whose introduction is credited to Sir James Inman (1776–1859), was utilized by navigators in the seventeenth and eighteenth centuries (Shylaja 2015).

The application of this straightaway gives the angular displacement of the moon or any object in the sky. The general formula used

$$\cos s = \sin\alpha 1 \sin\alpha 2 + \cos\delta 1 \cos\delta 2 \cos\Delta\alpha \quad (3)$$

can be replaced with

$$\text{hav } s = \text{hav } \Delta\delta + \cos\delta 1 \cos\delta 2 \text{ hav } \Delta\alpha. \quad (4)$$

This was suggested as a possible alternative for quick deductions as recently as in 1984 and it was strongly recommended that the method be reintroduced in textbooks (Sinnott 1984: 159).

Sawāī Jai Singh was a very meticulous observer as depicted by the various instruments he constructed at Varanasi, Delhi, Jaipur and Ujjain (the one in Mathurā is lost) which we know as Jantar Mantar. The instruments in his collection depict a variety of techniques. Thus, the simple instruments used for measuring altitude and azimuth were converted to the required coordinate system by replicating the night sky with grid. He was gifted with a telescope and used it for observing the satellites of Jupiter. He proceeded with the construction of massive instruments, since he believed that the accuracies can be achieved with massive structures (Sharma 1995). The tabletop instruments in his collection suggest that angular separations were measured fairly accurately.

Colonial Period

Cintāmaṇi Ragūnāthācārī (CR) was a meticulous observer and was quite well acquainted with the European methods of observations since he was working at the Madras Observatory. He participated in observations of stars, planets and eclipses (Shylaja 2012). He is the first Indian credited with the discovery of a variable star. He was able to compare the two seemingly different methods of planetary position computations (Indian and European). He wrote a monograph to educate local astronomers about the need for observations of the rare event – the transit of Venus – in 1874. He lists the timings of the transit for different places (*fig*. 19.3). Notice that the onset of the event is given in terms of the shadow length of the standard length of the gnomon (12''). CR refers to many contemporary astronomers and his correspondence with them. It should be interesting to search for the works by them. Some names are known like Bāpū Deva Śāstrī, other names are Śrīnivāsa Dīkṣita, Vaidyanātha Dīkṣita, Tolappar (he composed the *Śuddhi Vilocana*).

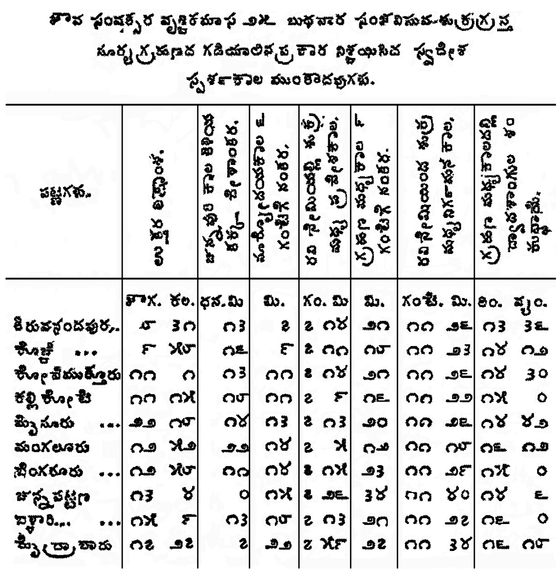


fig. 19.3: The table of timings of the transit of Venus of 1874, presented in terms of the shadow lengths of 12'' gnomon (last column)

CR urges people to take a look at the transit of Venus which occurred in November 1874. As a precursor to this he suggests the occultation of Venus by the moon on 12 November 1874. Interestingly, this happened during the day time. Another event he recommends is the conjunction of Mars and Jupiter on 16 December 1874.

In this context he had to coin new terms. For transit, the word generally used is *samāgama*. The title of the book itself is *Śukragrasta sūrya-grahaṇa*, equivalent for the word transit.

Here are the other words:

Lunar occultation	*candrachādana*
Grazing occultation	*sandigdha grahaṇa*
Meridian circle	*ardha-cakra*
Altazimuth circles	*digunnati-cakra*
Equatorial telescopes	*viṣuvadapekṣa*
Heliographic chronometer	*sarvato vedhakayantra*
Magnitude	*jyotiparimāṇa* (*sthūlatva*)
	jyotiparimāṇa nirṇāyakayantra??
Photographic apparatus chronometer?	*rūpagrāhaka yantra*
Telescope with coronograph arrangement?	*mukurānvita nalikāyantra*
Sidereal clock	*nakṣatra sāvana ghaṭikāyantra*
Barometer	*kuyāvu nirṇāyakayantra*
Thermometer	*śītoṣṇa nirṇāyakayantra*

Another astronomer of the same era who was well equipped with observations was Sāmanta Candraśekhara of Odisha. He had built all the instruments of the Siddhāntic texts. Unfortunately his work also does not describe the details of the methods of observations. It appears that he was not keeping in touch with the Arabic nomenclature of stars and instruments as well. Interestingly, he identifies Prajāpati with the constellation Orion.

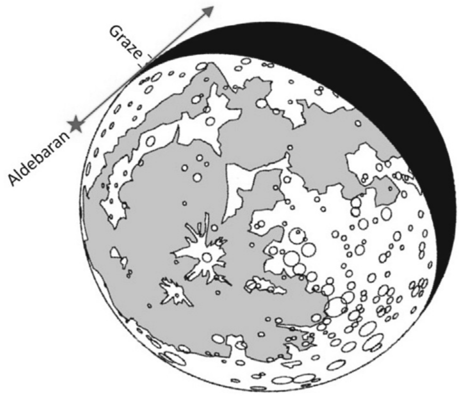


fig. 19.4: Lunar occultation definition of grazing occultation (*sandigdha grahaṇa*)

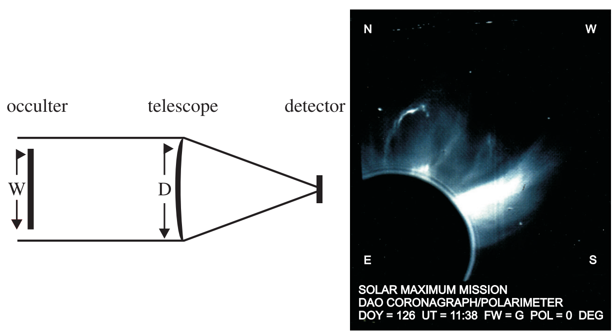


fig. 19.5: A coronograph blocks the central bright photosphere so that only the corona of the sun can be recorded (*mukurānvita nalikāyantra*)

We need to study the texts of early nineteenth century and twentieth century because in the process, some very interesting results emerged. One of them is the development of texts in regional languages. As pointed out earlier, CR put in great efforts to persuade Siddhāntic astronomers to utilize the modern gadgets like telescopes for accurate measurements. He wrote in Kannada, Persian/Urdu and Tamil (but Tamil text is not available). Others who read the English books tried to translate them and compare with the texts that were known to local people. There are at least three books in Kannada written prior to 1900 and at least one in

Bengali. The Bengali book got translated back to English since it included more of Hindu astronomy (Mukherji 1905). This book has coined the names of constellations; here are some examples:

Cassiopeia	Kāśyapīya
Sirius	Divyāśvan
Procyon	Saramā
Hercules	Bhīṣma
Perseus	Paraśurāma
Centaurus	Mahiṣāsura

There was already confusion in the names with the introduction of astrolabes for example:

Perseus as	*nṛpārśva, manuṣyapārśva*
Cygnus as	*pakṣī, samudrapakṣī*
Pegasus as	*haya* and *turaga*
Centaurus as	*kinnara* and *narāśva*

As you may be aware, the Hindi textbooks use the name Varuṇa for Uranus; this is quite confusing since a newly discovered asteroid has been named Varuṇa.

List of New Terms

It has been possible to list many synonyms and new words coined as part of the evolutionary processes. These include technical terms, names of stars and constellations. Some of these names have entered into non-astronomical texts also.

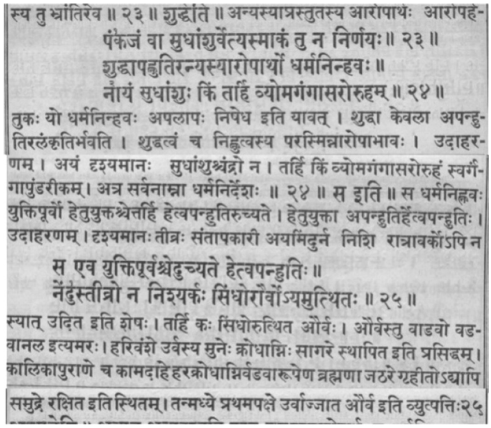
Aberration of light	*jyotirbhrama/tejobhrama*
Acceleration	*tvaraṇa/vegotkakrṣa*
Autumnal equinox / vernal equinox	*jalaviṣuva/mahāviṣuva*
Day circle	*dyurātravṛtta/dyuvṛtta/ahorātravṛtta*
Direct motion	*ṛjugati*
Ecliptic	*ravi mārga*
Eastern hemisphere	*pūrva-kapāla*

Western hemisphere	*paścima-kapāla*
Libration	*tolana*
Museum	*durlabha-vastuśālā*
Meteoric stone	*dhiṣṇya*
Nadir	*adhahsvastika*
Observatory	*dṛgāgāra*
Penumbra	*pūrṇavacchāyā*
Perturbation	*tuyta*
Ring of Saturn	*kaṭaka*
Zenith	*svastika/ākāśamadhya/khamadhya/ ūrdhvasvastika/nabhomadhya*
Zodiac	*rāśi-cakra, jyotiṣa-cakra*

When Was Milky Way Called Ākāśagaṅgā?

The Milky Way is such a mesmerizing sight in the sky that it did charm poets and artists in India as it did elsewhere. It has *nakṣatrapatha*, *surapatha* and similar names in Vālmīki's *Rāmāyaṇa* and Kālidāsa's *Raghuvaṁśa*. However, today it is known to us as *ākāśagaṅgā*. This name first appears in a Sanskrit text on *alaṁkāra*s by Appayya Dīkṣita (*fig*. 19.6). He mentions a lotus in *vyomagaṅgā*. Therefore, we may assume that by seventeenth century the influence of Persian names were recognizable.

We find several names like *śiśumāra* and *matsyodara*, which are not translations of Persian names. The identification and origin of these names are yet to be sorted out.



स्य तु भ्रांतिरेव ॥ २३ ॥ शुद्धेति ॥ अन्यस्याप्रस्तुतस्य आरोपार्थः आरोपहे-

पंकजं वा सुधांशुर्वेत्यस्माकं तु न निर्णयः॥ २३ ॥
शुद्धापह्नुतिरन्यस्यारोपार्थो धर्मनिन्हवः ॥
नायं सुधांशुः किं तर्हि व्योमगंगासरोरुहम् ॥ २४ ॥

तुकः यो धर्मनिन्हवः अपलापः निपेध इति यावत् । शुद्धा केवला अपन्हु-
तिरलंकृतिर्भवति । शुद्धत्वं च निह्नुत्वस्य परस्मिन्नारोपाभावः । उदाहर-
णम् । अयं दृश्यमानः सुधांशुश्चंद्रो न । तर्हि किं व्योमगंगासरोरुहं स्वर्ग-
गापुंडरीकम्। अत्र सर्वनाम्ना धर्मनिर्देशः ॥ २४ ॥ स इति ॥ स धर्मनिह्नवः
युक्तिपूर्वो हेतुयुक्तश्चेत्तर्हि हेत्वपन्हुतिरुच्यते । हेतुयुक्ता अपन्हुतिर्हेत्वपन्हुतिः ।
उदाहरणम् । दृश्यमानः तीव्रः संतापकारी अयमिंदुर्न निशि रात्रावर्कोऽपि न

स एव युक्तिपूर्वश्चेदुच्यते हेत्वपन्हुतिः ॥
नेंदुस्तीव्रो न निश्यर्कः सिंधोरौर्वोऽयमुत्थितः ॥ २५ ॥

स्यात् उदित इति शेषः । तर्हि कः सिंधोरुत्थित और्वः । और्वस्तु वाडवो वड-
वानल इत्यमरः । हरिवंशे उर्वस्य मुनेः क्रोधाग्निः सागरे स्थापित इति प्रसिद्धम् ।
कालिकापुराणे च कामदाहे हरक्रोधाग्निर्वडवारूपेण ब्रह्मणा जठरे गृहीतोऽद्यापि
समुद्रे रक्षित इति स्थितम् । तन्मध्ये प्रथमपक्षे उर्वाज्जात और्व इति व्युत्पत्तिः२५

Kuvalayananda of Appayya Diksita, Karnataka Samskrit University

fig. 19.6: The earliest reference to *vyomagaṅgā*

Conclusion

The study aims at understanding the finer details on the instruments and observational procedures. In an attempt to extract the observational procedures several interesting applications have been identified. Some terms like *apacchāyā*, when properly interpreted, reveal finer details of measurement. This also throws light on some new unknown words perhaps coined for the need. Transit of Venus is one such example. This is especially true in the texts of late nineteenth century when the usage of telescopes was being introduced. A list of such new words has been presented and discussed.

References

Mukherji, K., 1905, *Popular Hindu Astronomy*, Calcutta: Hare Press, (reprinted in 1969 by Nirmal Mukherjea).

Ohashi, Y., 1994, "Astronomical Instruments in Classical Siddhāntas", *Indian J. Hist. Sci.*, 29(2).

———, 1997, "Early History of Astrolabes in India", *Indian J. Hist. Sci.*, 32(3): 199-295.

Online version of *Kuvalayānanda* published by Karnataka Samskrit University, Bangalore.

Rao, S. Balachandra and S.K. Uma, 2006, *Grahalāghava*, New Delhi: INSA Publication.

Sarma, S.R., 2018, online catalogue of astrolabes, https://srsarma.in/catalogue.php

Sharma, V.N., 1995, *Sawai Jai Singh and His Astronomy*, New Delhi: Motilal Banarsidass.

Shylaja, B.S., 2012, *Chintāmani Ragoonathāchari and Contemporary Indian Astronomy*, Bangaluru: Navakarnataka Publications.

———, 2015, "From Navigation to Star Hopping: Forgotten Formulae", *Resonance,* April, pp. 352-59.

Sinnott, R.W., 1984, "Virtues of the Haversine", *Sky and Telescope*, 68(2).

Venketeswara, R. Pai and B.S. Shylaja, 2016, "Measurements of Co-ordinates of Nakṣatras in Indian Astronomy", *Current Science*, 111(a).

20

Mahājyānayanaprakāraḥ

Infinite Series for the Sine and Cosine Functions in the Kerala Works

G. Raja Rajeswari
M.S. Sriram

Abstract: It is well known that the infinite series expansion for the sine and cosine functions were first discussed in the Kerala works on astronomy and mathematics and are invariably ascribed to Mādhava of Saṅgrāmagrāma (fourteenth century CE). The full proofs of these are to be found in the *Gaṇitayuktibhāṣā* of Jyeṣṭhadeva (composed around 1530 CE). However, there is a Kerala work called the *Mahājyānayanaprakāraḥ,* which describes the infinite series for the *jyā* ($R \sin \theta$) and the *śarā* ($R(1 - \cos \theta)$) and provides a shorter derivation of them. This was discussed in a paper by David Gold and David Pingree in 1991. However, that paper did not explain the derivation of the infinite series in the manuscript. In this paper we provide the derivation based on the *upapatti* provided by the author of the manuscript.

Keywords: Infinite series, *jyā*, Kerala school, Mādhava, *śarā*, derivation.

Introduction

THE Kerala school of astronomy and mathematics (fourteenth–nineteenth centuries) is well known for its pioneering work on mathematical analysis, especially the discovery of the infinite series for π and also sine and cosine functions. In modern notation (Sarma 1972), the infinte series for the latter are:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \ldots \quad (1)$$

$$1 - \cos\theta = \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \frac{\theta^8}{8!} \ldots. \quad (2)$$

They do not appear in any of the discovered works of Mādhava, the founder of the school, but are invariably ascribed to him by the later astronomer-mathematicians of the school like Jyeṣṭhadeva and Śaṅkara Varier. The *Gaṇitayuktibhāṣā* of Jyeṣṭhadeva (*c*.1530) is perhaps the first work to give the detailed derivation of all the infinte series (Sarma 2008).

K.V. Sarma was perhaps the first to notice a manuscript in Sanskrit named the *Mahājyānayānaprakāraḥ* which describes the infinite series for the sine and cosine functions and also gives the *upapatti* (derivation) for the same, though he did not discuss the manuscript in detail. This manuscript was available in the India Office Library in London.[1] A handwritten version of the manuscript was prepared by K.V. Sarma and it is available in the Prof. K.V. Sarma Research Foundation, Adyar , Chennai. K.V. Sarma ascribed the authorship of the manuscript to Mādhava himself.

In a paper published in 1991, David Gold and David Pingree gave a full edition of this manuscript, and also provided the translation. However they did not provide any explanation of the derivation of the infinte series, as described by the author. Gold and Pingree argued that the author could not have been Mādhava, but definitely from the "Mādhava school".

One of the authors of the present paper (G. Rajarajeswari) had worked on the manuscript for her MPhil thesis submitted to the University of Madras in August 2010. In that thesis, the manuscript had been translated into English afresh, and detailed explanatory notes had been provided. The present paper is essentially a summary of the thesis.

[1] Ff. 12-16 of a manuscript, Burnell 17e, India Office Library, London.

In this paper, we have explained the derivation of the series for $R \sin \theta$ and $R (1 - \cos \theta)$, completely based on the description of the derivation in the work. This derivation is very similar to the one in the *Gaṇitayuktibhāṣā*, but differs from it in some respects. Both the derivations are based on the iterative solution of the discrete version of the equ ations:

$$\sin\theta = \int_0^\theta \cos\theta' d\theta' = \theta - \int_0^\theta (1-\cos\theta') d\theta',$$

$$1-\cos\theta = \int_0^\theta \sin\theta' d\theta'.$$

Description of the Series for the R sine and the Numerical R sine Values

The manuscript has three sections, viz. the explanation of the series, the method to derive the numerical sine values and the derivation for both the sine and cosine series.

The author begins with the description of the series for R sine θ:

निहत्य चापवर्गेण चापं तत्तत्फलानि च।
हरेत् समूलयुग्वर्गैस्त्रिज्यावर्गहतैः क्रमात्।।

चापं फलानि चाधोऽधो न्यस्योपर्युपरि त्यजेत्।
जीवाप्त्यै संग्रहोऽस्यैव विद्वानित्यादिना कृतः।।

Mulitiply the arc ($r\theta$) and the [successive] results by the square of the arc, divide (each of the above numerators) by the squares of the successive even numbers increased by that number and multiplied by the square of the radius in order. Place the arc and the successive results so obtained one below the other and subtract each from the one above. These together give the *jīvā*, as collected together in the verse beginning with *vidvān*, etc.

Hence,

$$R\sin\theta = (R\theta) - \left[\frac{(R\theta)(R\theta)^2}{(2^2+2)R^2}\right.$$

$$-\left[\frac{(R\theta)(R\theta)^2(R\theta)^2}{(2^2+2)(4^4+4)R^2R^2}\right.$$

$$-\left[\frac{(R\theta)(R\theta)^2(R\theta)^2(R\theta)^2}{(2^2+2)(4^4+4)(6^6+6)R^2R^2R^2} - \ldots\right.$$

Here R is the *trijyā* (radius) of a circle whose circumference is 21600. In fact,

$$R \approx 3437 + \frac{44}{60} + \frac{48}{3600}\ .$$

In modern notation,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

He also explains how we can get numerical values of $R \sin \theta$ for any θ, using the method given by Śaṅkara Varier and others.

Derivation of the Series

OBTAINING THE COUPLED EQUATIONS
FOR THE JĪVĀ, R sin θ AND THE ŚARĀ, R(1 – COS θ)

We provide the essential steps in the author's derivation of the series. First, he obtains the *jīvā, R* sin θ as the sum of the intermediate *koṭi*'s and the Ś*arā*, $R(1 - \cos\theta)$ as a sum of the intermediate *jīvā*s.

> निहत्य चापवर्गेण इत्यादिना इष्टचापस्य ज्यानयने कीदृश्युपपत्तिः। तत्प्रदर्शनाय वृत्तमालिख्य मातृपितृरेखां च कुर्यात्।
>
> What is the proof for finding the *jīvā* for the desired arc, as given by the *śloka*, *nihatya cāpavargeṇa* . ... For demonstrating that result, let a circle be drawn and let the "east–west and the north–south" (*Y* and *X* axis) (*mātṛ-pitṛ-rekhā*) be marked.

The author begins thus:

ES is a quadrant of a circle of radius *R*. *OE* and *OS* are the east–west and north–south lines.

Consider the

$$\text{arc } EC = R\theta.$$

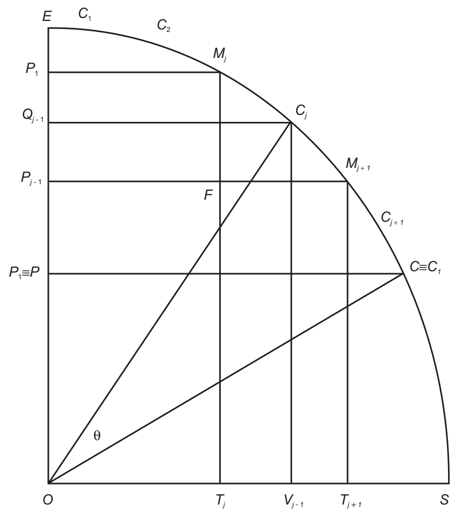
This is divided into *n* equal arc bits (where *n* is a large number):

$$EC_1 = C_1C_2 = \ldots\ C_jC_{j+1} \ldots = C_{n-1}\,C_n \equiv R\theta/n = \alpha$$

and

$$EC_j = R\theta_j = R\,\frac{j\theta}{n} = Rj\alpha.$$

Draw C_jP_j parallel to OS and C_jT_j parallel to OE:



$C_jP_j = R \sin \theta_j = R \sin (\alpha j)$.

C_n is the same as C and P_n is the same as P.

$C_nP_n = CP = R \sin \theta$.

$B_j = C_jP_j = R \sin (\alpha \cdot j)$ is the *bhujā* corresponding to the arc EC_j and $K_j = C_jT_j = OP_j\ R \cos (\alpha \cdot j)$ is the *koṭi* corresponding to the arc EC_j. Let M_{j+1} be the midpoint of the arc C_jC_{j+1}. Then,

$B_{j+1/2} = M_{j+1}Q_{j+1} = R \sin (\alpha \cdot (j + 1/2))$ is the *bhujā* corresponding to the arc EM_{j+1} , and $K_{j+1/2} = M_{j+1}V_{j+1} = OQ_{j+1} = R \cos (\alpha \cdot (j + ½))$ is the *koṭi* corresponding to the arc EM_{j+1}.

Now the *bhujā khaṇḍa* is the R sine difference or *bhujā khaṇḍa* $= B_{j+1} - B_j = R \sin (\alpha \cdot (j + 1)) - R \sin (\alpha \cdot j) = FC_{j+1}$.

The *samasta jyā* of the arc-bit C_jC_{j+1} is the full-chord, C_jC_{j+1}. *Koṭi khaṇḍa* $= K_j - K_{j+1} = C_jT_j - C_{j+1}\ T_{j+1} = R \cos (\alpha j) - R \cos (\alpha(j + 1)) = FC_j$.

The author considers the two right triangles, $C_{j+1}FC_j$ and $OQ_{j+1}\ M_{j+1}$. He says:

तत्रतत्क्षेत्रद्वस्य तुल्याकारान्निर्णयः

The two geometrical figures are similar.

He explains that they are similar as OM_{j+1} is perpendicular to C_jC_{j+1} and C_{j+1}. F is perpendicular to T_jF and hence to OQ_{j+1}.

Then the author says:

तदानीं तस्य चापखण्डस्य समस्तज्यां भुजकोटिभ्यां पृथङ्निहत्य त्रिज्यया द्वे अपि विभज्य लब्धे कोटिखण्डं भुजाखण्डं च भवतः

Then when the full-chord of the arc-bit (*samasta jyā*) is multiplied by the *R* sine and *R* cosine (of the desired arc) and divided by the Radius (*R*) separately, *R* cosine – difference (*koṭi khaṇḍa*) and *R* sine – difference (*bhujā khaṇḍa*) of the arc-bit are the results.

So, according to him:

$$\textit{Bhujā khaṇḍa} = \frac{\textit{Koṭi} \times \textit{Samasta jyā}}{R},$$

and

$$\textit{Koṭi khaṇḍa} = \frac{\textit{Bhujā} \times \textit{Samasta jyā}}{R}.$$

This can be understood as follows:

Because of the C similarity of the triangles, we have

$$\frac{C_jF}{C_jC_{j+1}} = \frac{OQ_{j+1}}{OM_{j+1}}.$$

Here,

$C_jF = B_{j+1} - B_j$ = *Bhujā khaṇḍa*.

$C_j\,C_{j+1}$ = Chord = *Samasta jyā* ≈ arc $C_jC_{j+1} = \alpha$.

OQ_{j+1} is the *koṭi* at the mid-point M_{j+1}. $OM_{j+1} = R$. Hence,

$$B_{j+1} - B_j = \frac{\alpha \cdot K_{j+½}}{R}.$$

Similarly,

$$K_j - K_{j+1} = \frac{\alpha B_{j+½}}{R}.$$

Then the author says:

एवं खण्डमध्यप्रसृतानां कोटिज्यानां योगेन खण्डं निहत्य त्रिज्यया विभज्य लब्दमिष्टे जीवा भवति।

When the arc-bit is multiplied by the sum of *koṭi jyā*s proceeding from the mid-point of all the arc-bits and divided by the radius R, what results is the *jīvā*.

We explain this below:

Now, the R sine of the arc EC is $CP = R \sin\theta$:

$CP = R\sin\theta = P_n C_n - 0 = B_n - B_0$.

We write this as,

$$R\sin\theta = B_n - B_0 = (B_n - B_{n-1}) + (B_{n-1} - B_{n-2}) + ... + (B_1 - B_0).$$

Using the relation between the *bhujā khaṇḍa* and the mid-point (*koṭi*)

$$\textit{jīvā} = R\sin\theta = \frac{\alpha}{R}\sum_{j=0}^{n-1} K_{j+\frac{1}{2}}.$$

Now,

$$S = \textit{Śarā} = R - R\cos\theta = R(1-\cos\theta).$$
$$S_{j+½} = R - K_{j+½} = R - V_{j+1}M_{j+1} = OE - OQ_{j+1} = EQ_{j+1}.$$

Now,

$$K_{j+½} = R - S_{j+½}$$

Hence,

$$R\sin\theta = \frac{\alpha}{R}\sum_{j=0}^{n-1}(R - S_{j+1/2}), \text{or}$$

$$R\sin\theta = \frac{\alpha}{R}\cdot R\cdot n - \frac{\alpha}{R}\sum S_{j+1/2}$$

As $\quad R\theta = \alpha n.$

$$\therefore R\sin\theta = R\theta - \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} S_{j+1/2}.$$

This is the discret *R*e version of

$$R\sin\theta = R\theta - \int_0^\theta (1-\cos\theta')\,d\theta',$$

where $\frac{\alpha}{R}$ corresponds to $d\theta'$.

Similarly, we find

$$R(1-\cos\theta) = S = \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} B_{j+1/2}..$$

This is the discrete version of

$$R(1-\cos\theta) = \int_0^\theta \sin\theta'\,d\theta'.$$

Iterative technique for solving the coupled equations for R sin θ and R(1 – cos θ)

We have the equations

$$R\sin\theta = R\theta - \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} S_{j+1/2}.$$

and

$$R(1-\cos\theta) = S = \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} B_{j+1/2}.$$

The author uses an ingenious iterative technique to solve these equations and obtain the infinite series.

Zeroth Approximation

Here all *śarā*s are taken to be zero: $S_j = S_{j+1} = 0$.

Then,

$$R \sin \theta = R \theta.$$

and

$$S = R(1 - \cos \theta) = 0.$$

First Approximation

In this approximation, in the expression for S, which is a sum of the *bhujā*s, the *bhujā*s are taken to be the arcs themselves.

So,

$$S = R(1-\cos\theta) = \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} B_{j+1/2} \approx \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} \mathrm{Arc}\left(EC_{j+1/2}\right)$$
$$\approx \left(\frac{\alpha}{R}\right)\sum_{j=0}^{n-1} \mathrm{Arc}\left(EC_j\right).$$

Now, Arc $EC_j = j\alpha$.

$$\therefore S = \frac{\alpha}{R}\cdot\alpha\sum_{j=0}^{n-1} j$$
$$= \frac{\alpha}{R}\cdot\alpha\cdot n\frac{n-1}{2}$$
$$\approx \frac{\alpha^2 n^2}{2R} = \frac{(R\theta)^2}{2R}.$$
$$\therefore S = R(1-\cos\theta) = \frac{(R\theta)^2}{2R}.$$

In the words of the author:

खण्डमध्यादि वा खण्डादि वा खण्डोत्तरं इष्टचापान्तरं यत् संकलितं तस्मिन् खण्डगुणिते सति इष्टचापवर्गार्धं सम्पद्यते। अत्र यत्किञ्चिन्न्यूनातिरिको दृश्यते सवर्णस्याल्पत्वे कृते मण्डूकशिशूनां लाङ्गूलवत् लुप्यमाना तस्मिन्नेव निमज्जति।

When the arcs from the beginning of the desired arc to the middle or beginning or ending points of the arc-bits are summed over, and multiplied by the arc-bit then, half the square of the desired arc is obtained. Whatever appears as the deficiency or the excess will get eliminated just as the "tail of a frog's new offspring disappears in itself".

Here, we have used $\sum_{j=0}^{n-1} j \approx \frac{n^2}{2}$.

This is the discrete equivalent of

$$\int x dx = \frac{x^2}{2}.$$

In this approximation, summation is done by taking n to be very large, and the limit is taken properly. So, actually, "calculus" type of ideas are used.

Consider now the first approximation for $R \sin \theta$.

The author says:

जीवानयने तु खण्डमध्यप्रसृताभिः कोटिभिः खण्डस्य गुणने कर्तव्येऽपि कोटिज्यान्तरभूतैः शरखण्डं हत्वा त्रिज्यया विभज्य लब्धमिष्टचापाद्विशोध्यम् शिष्टं इष्टचापज्या भवति।

अथवा शरैक्येन खण्डं गुणन्यन् संकलितांशभूतानां ऐक्यं तत्संकलितसंकलितमेव। अतः संकलितज्ञानाय त्रिज्यया विभक्तमिष्टचापार्धमिष्टचापेन निहत्य त्रिसंख्येन विभज्य पुनः त्रिज्यया च हरेत्। लब्धं ज्यानयनाय चापाच्छोध्यं च भवति।

For finding the *jīvā*, multiply the *cāpa khaṇḍa* with the *koṭi*, corresponding to the middle of the arc-bit. Further, multiply by the *śarakhaṇḍam* which is the difference between the *koṭi jyā*s, and divide it by the radius R. Subtract the obtained result by the arbitrary arc. What results is the *jīvā.*

In other words, the product of the sum of the *śara*s multiplied by the arc-bit is the sum of the sums only. Therefore, to know the summation, half of the square of arc is divided by the radius R, multiplied by the desired arc, divided by the number 3 and divided again by radius R. This result is to be subtracted from the desired arc to obtain the desired R sine (*jyā*).

We explain this below.

Now,

$$R\sin\theta = R\theta - \frac{\alpha}{R}\sum_{j=0}^{n-1} S_{j+1/2}$$
$$\approx R\theta - \frac{\alpha}{R}\sum_{j=0}^{n-1} S_j.$$

Now, $$S = \frac{\alpha}{R}\sum_{j=0}^{n-1} B_{j+1/2}.$$

This corresponds to the arc $R\theta = \alpha n$.

S_j corresponds to the arc $R\theta_j = \alpha j$. Hence,

$$S_j = \frac{\alpha}{R}\sum_{r=0}^{j-1} B_{r+1/2}.$$

Hence, the expression for $R \sin \theta$ is a sum of sums.

Now, $$B_{r+1/2} \approx \left(r + \frac{1}{2}\right)\alpha.$$

Hence, $$R\sin\theta \approx R\theta - \frac{\alpha^3}{R^2}\sum_{j=0}^{n-1}\sum_{r=0}^{j-1}(r + ½)$$
$$\approx R\theta - \frac{\alpha^3}{R^2}\sum_{j=0}^{n-1}\sum_{r=0}^{j}(r).$$

Now,

$$\sum_{j=0}^{n-1}\sum_{r=0}^{j}(r) = \sum_{j=0}^{n-1}\frac{j(j+1)}{2} = \frac{(n+1)n(n+1)}{1\cdot 2\cdot 3}.$$

This is the correct result for the double summation. In the large n limit, $(n - 1)n(n + 1)$ can be replaced by n^3. Then, using

$$\alpha n = R\theta$$

we have,

$$R\sin\theta \approx R\theta - \frac{\alpha^3 n^3}{R^2 1\cdot 2\cdot 3}$$
$$\approx R\theta - \frac{(R\theta)^3}{R^2 1\cdot 2\cdot 3}.$$

Now, consider the *śara* expression:

$$S = \frac{\alpha}{R} \sum B_{j+1/2}.$$

The author notes the following:

पूर्वं शरानयनाय खण्डमध्यप्रसृतानां जीवानां योगेन खण्डेन गुणनीयेपि तासामज्ञातत्त्वाच्चापानामेव योगः कृतः।

Earlier, for obtaining the *śara*, though the sum of *jīvā*s that proceeded from the middle of each part was to be considered; because the *jīvā*s were not known, the summation of the arcs itself was considered (that is, the *jīvā*s were taken to be the arcs themselves in the first approximation).

इदानीं ज्यानयनाय चापादियच्छोध्यमिति जातम्। अतस्तेषामैक्यमाऽनेयम्। ज्ञातानि ज्याचापान्तरानि तु संकलितैक्यांशीभूतानि । तेषां योगस्यानयनाय ज्ञातज्याचापान्तं इष्टचापेने निहत्य चतुस्संख्येन हर्तव्यम्। तत्पुनः त्रिज्यया विभज्य पूर्वानीताच्छरात् शोध्यत्वं च।

Now, in order to obtain the *jyā*, we know that "this" is the measure that needs to be subtracted from *cāpa*. Hence we need to find the sum of them. The differences of the *jīvā* and arc [at each *khaṇḍa madhya*] known, have become part of the sum of the sums.

To find their sum, multiply the difference of *jyā* and *cāpa* (which is known) by the arbitrary arc, and divide it by number 4. Again divide it by Radius *R* and subtract it from the *śara* obtained before.

We explain thSis below:

$$S = \frac{\alpha}{R} \sum B_{j+½}$$

$$= \frac{\alpha}{R}\left(\sum_{j=0}^{n-1} (j+½)\alpha - \sum_{j=0}^{n-1} \frac{\{(j+½)\alpha\}^3}{R^2 \cdot 2 \cdot 3} \right).$$

$$\sum_{j=0}^{n-1} (j+½) \approx \frac{n^2}{2}.$$

This is the discrete equivalent of $\int x dx = \frac{x^2}{2}$.

$$\sum_{j=0}^{n-1}(j+½)^3 \approx \sum_{j=0}^{n-1} j^3 \approx \frac{n^4}{4}.$$

This is the discrete equivalent of $\int x^3 dx = \dfrac{x^4}{4}$.

As $\alpha n = R\theta$,

$$S = \frac{\alpha^2 \cdot n^2}{2R} - \frac{\alpha^4 n^4}{R^3 4!}$$
$$= \frac{(R\theta)^2}{2R} - \frac{(R\theta)^4}{R^3 4!}.$$

Now, this expression for the *śara* has to be used to find the next approximation for $R \sin \theta$.

In the words of the author:

पुनः ज्याशोधनाय शराच्छोधितस्य संकलितं कार्यम्। तदर्थमानेयं तृतीयफलम्। इष्टचापेन हत्वा पञ्चसंख्येन विभज्य त्रिज्यया हर्तव्यम्।

> Again to find the correction to the *jyā*, we have to find the sum of the corrections to the *śara* (at each *khaṇḍa madhya*). For that purpose, find the third result and multiply it by the arbitrary arc, and divide it by number 5, multiplied by the radius *R*.

We explain this below:

$$R \sin = R\theta - \frac{\alpha}{R}\sum_{j=0}^{n-1} S_j$$
$$= R\theta - \frac{\alpha}{R}\sum\left\{\frac{\alpha^2 j^2}{2R} - \frac{\alpha^4 j^4}{R^3 4!}\right\}$$
$$\approx R\theta - \frac{\alpha^3 n^3}{R^2 \cdot 2 \cdot 3} + \frac{\alpha^5 j^5}{R^4 5!},$$

where we have used

$\sum_{j=0}^{n-1} j^4 \approx \dfrac{n^4}{5}$ for large n and $\alpha n = R\ \theta$.

$$\therefore R \sin \theta = R\theta - \frac{(R\theta)^3}{R^2 \cdot 3!} + \frac{(R\theta)^5}{R^4 \cdot 5!}.$$

The author says:

एवमुत्तरोत्तरफलानि नेयानि।

Bring out the remaining results in the same manner.

So, this procedure should be continued to obtain the successive corrections to the *jīvā*, $R \sin \theta$ and the *śara*, $R\,(1 - \cos \theta)$.

The infinite series for the *jīvā*, $R \sin \theta$ and *śara*, $R(1 - \cos \theta)$ are stated again by the author.

He notes that $n^2 + n = n(n + 1)$. So he states the series for $R \sin \theta$ and $R(1 - \cos \theta)$ in the modern form (apart from the appearance of R):

$$R \sin \theta = R\theta - \frac{R\theta^3}{2 \cdot 3 \cdot R^2} + \frac{R\theta^5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot R^4} - \ldots$$

$$R(1 - \cos \theta) = \frac{R\theta^2}{2 \cdot R^2} - \frac{R\theta^4}{2 \cdot 3 \cdot 4 \cdot R^4} + \ldots.$$

Concluding Remarks

It appears that the infinte series for π and the sine and cosine functions had become common knowledge among the astronomer-mathematicians of Kerala by the sixteenth century. The *Yuktibhāṣā* gives a detailed derivation for the same. In a short Kerala manuscript *Mahājyānayanaprakāraḥ*, the author states the infinite series and discusses the method due to Śaṅkara Varier and others to compute them. More importantly, he gives a simple and elegant derivation of the series for $R \sin \theta$ and $R(1 - \cos \theta)$. It is a compact version of the derivation in the *Yuktibhāṣā*. In this paper we have explained this derivation using the modern notation.

Acknowledgements

One of the authors (G. Rajarajeswari) is grateful to Professor P. Narasimhan, Head of the Department of Sanskrit, University of Madras, and also Professor M.D. Srinivas, Centre for Policy Studies, Chennai and Professor K. Ramasubramanian, IIT Bombay for suggesting many improvements in her MPhil thesis, on which this paper is based.

References

Gold, David and David Pingree, 1991, "A Hitherto Unknown Sanskrit Work Concerning Mādhava's Derivation of the Power Series for Sine and Cosine" *Historia Scientarium*, 42: 49-65.

Rajarajeswari, G., 2010, *Mahājyānayanaprakāraḥ of Mādhava of Saṅgamagrāma*, a critical edition with translation and notes, MPhil thesis, Department of Sanskrit, University of Madras.

Sarma, K.V., 1972, *A History of the Kerala School of Hindu Astronomy in Perspective*, Hoshiarpur: Vishveshvaranda Institute.

Sarma, K.V. (ed. and tr.), 2008, *Gaṇitayuktibhāṣā of Jyeṣṭhadeva*, with explanatory notes by K. Ramasubramanian, M.D. Srinivas and M.S. Sriram, 2 vols, New Delhi: Hindustan Book Agency (repr. Springer, 2009).

21

Lunar Eclipse Calculations in Tantrasaṁgraha (c.1500 CE)

D. Hannah Jerrin Thangam
R. Radhakrishnan
M.S. Sriram

Abstract : We discuss the calculations pertaining to a lunar eclipse in the celebrated Kerala work on astronomy, *Tantrasaṁgraha* (*c*.1500 CE). We outline the procedure for computing the middle of the eclipse, the half durations and the half durations of totality, using iterative processes. We illustrate the procedure by taking up the lunar eclipse which occurred on 27/28 July 2018. We compare the computed values based on *Tantrasaṁgraha*, with those obtained using the modern procedures and tabulated in modern almanacs like the Rāṣṭrīya Pañchāṅga, published by the India Meteorological Department. We also make the comparison for another recent eclipse on 7 August 2017. There is a very remarkable agreement between the tabulated values and those computed using the *Tantrasaṁgraha* procedure, for both the eclipses.

Introduction

INDIA had an unique, definitive and very significant tradition in astronomy right from the Vedic times (see for instance, Sen and Shukla 1985). Simplicity of the calculational procedure is a characteristic feature of the Indian astronomical tradition. This is particularly true of the computation of the planetary longitudes and latitudes. Even with such simplified procedures, the computed values are reasonably accurate. Consider for instance, *Tantrasaṁgraha* (c.1500 CE), the celebrated Kerala work on astronomy (Sarma 1977 ; Ramasubramanian and Sriram 2011). The computed value of the moon's longitude in *Tantrasaṁgraha* is correct up to a degree, on the average, even for modern dates (Sriram and Ramasubramanian 1994).

The physical variables associated with the lunar and solar eclipses (like the instant of conjunction or opposition, half durations of the eclipse, etc.) are very sensitive to the parameters associated with the sun and the moon, and the particular procedure for computations. They are critically tested during eclipses. In fact, it was standard Indian practice to revise the parameter based on eclipse observations.

Parallax does not play a role in lunar eclipse calculations, whereas it has a very significant effect on the occurence of a solar eclipse and its progress. Correspondingly, the calculations are that much harder for a solar eclipse. In this article, we confine our attention to the computation of a lunar eclipse in the celebrated text *Tantrasaṁgraha* of Nīlakaṇṭha Somayājī (*c*.1500 CE).

A lunar eclipse occurs when the earth's shadow blocks the the sun's light, which otherwise reflects off the moon. There are three types – total, partial and penumbral – with the most dramatic being a total lunar eclipse, in which the earth's shadow completely covers the moon. A lunar eclipse can occur only at full moon. A total lunar eclipse can happen only when the sun, the earth and the moon are perfectly lined up (at least for a short time interval) – anything less than perfection creates a partial lunar eclipse, or no eclipse at all. As the moon's orbit around the earth is inclined to the earth's orbit around the sun, an eclipse doesn't occur at every full moon; a total lunar eclipse is even rarer, as the "perfect" alignment is even rarer.

Lunar Eclipse Computations in Tantrasaṁgraha

NUMBER OF PLANETARY REVOLUTIONS IN A MAHĀYUGA AND AHARGAṆA

We have to first find the time of conjunction of the moon and the earth's shadow, or the instant when the sun and the moon are in opposition as viewed from the earth. To determine this instant, the first step is to find the mean longitudes of the sun, the moon, the latter's node and also its apogee. The mean longitude of any object can be determined using the mean rate of motion of the object, and the *ahargaṇa*, which is the number or the count of days from an epoch. The mean rate of motion is found from the number of revolutions made by the object in a *mahāyuga* of 43,20,000 years, and the number of civil days in a *mahāyuga*.

All the calculations in the ancient Indian texts are in a geocentric framework, in which the sun also revolves around the earth. The following table (Table 21.1) gives the number of revolutions completed by the sun, the moon, its apogee and its node in a *mahāyuga* in *Tantrasaṁgraha*. These values are the same as in *Āryabhaṭīya* of Āryabhaṭa (*c.*499 CE), the first available Siddhāntic text in the Indian tradition (Shukla and Sarma 1976).

According to both *Āryabhaṭīya* and *Tantrasaṁgraha*, the number of civil days in a *mahāyuga* is 1,577,917,500 days.

Mean rates of Motion of the Sun, Moon, Moon's apogee and Moon's Node

If N is the number of revolutions of an object in a *mahāyuga*, its mean rate of motion in degrees per day is given by

$$\text{Mean rate of motion (degrees per day)} = \left(\frac{N}{1,577,917,500}\right) \times 360°.$$

Table 21.1: The Number of Revolutions Completed by the Planets in a Mahāyuga of 4,320,000 years in Tantrasaṁgraha

Planet	*No.of Resolutions* (N)
Sun	4,320,000
Moon	57,753,320
Moon's apogee	488,122
Moon's node	232,300

The *Tantrasaṁgraha* values of the mean rates of motion are presented in Table 21.2.

Table 21.2: Mean Rates of Motion

No.	*Planet*	*Mean Rate of Motion (in degrees/day)*
1.	Sun	0.985602859
2.	Moon	13.17635124
3.	Moon's Apogee	0.111364453
4.	Moon's node	– 0.052998968

Mean Longitudes of the Sun, Moon and Moon's Node

It is straightforward to obtain the mean longitudes of the planets from the *ahargaṇa*. Let *A* be the *ahargaṇa* and *N* the number of revolutions completed by the planet in a *mahāyuga*. Then, the number of revolutions including the fractional part covered by the planet since the epoch, till the mean sunrise (local time of 6 a.m.) at the traditional standard Indian meridian, namely, Ujjain, is given by:

$$n = \frac{A \times N}{1{,}577{,}917{,}500}.$$

We have to take the epochal value of the mean longitude, denoted by θ (epoch) , also into account. As the integral multiples of 360° are not taken into account in the longitudes, the mean longitude corresponding to an *ahargaṇa*, *A* is given by:

$$\theta_0 = \left(\frac{A \times N}{1{,}577{,}917{,}500}\right) f \times 360° + \theta(\text{epoch})$$

In *Tantrasaṁgraha*, the epoch is the Kali-Yuga beginning, which corresponds to the mean sunrise at Ujjain on 18 February 3102 BCE. The mean longitudes of the objects relevant for a lunar eclipse corrsponding to any *ahargaṇa*, *A* are presented in the Table 21.3:

Table 21.3: Mean Longitudes of the Planets for an *Ahargaṇa*, *A* at Mean Sunrise at Ujjain

Planet	θ (*Epoch*)	*Mean longitude*, θ_0 *for an ahargaṇa, A*
Sun	0°0'0"	$\theta_{0\,\text{Sun}} = \left(\dfrac{A \times 43{,}20{,}000}{1{,}577{,}917{,}500}\right)f \times 360^\circ$
Moon	4°45'46"	$\theta_{0\,\text{Moon}} = \left(\dfrac{A \times 57{,}753{,}320}{1{,}577{,}917{,}500}\right)f \times 360^\circ + 4^\circ 45' 46''$
Moon's apogee	119°17'5"	$\theta_{0\,\text{Moon' apogee}} = \left(\dfrac{A \times 488{,}122}{1{,}577{,}917{,}500}\right)f \times 360^\circ + 119^\circ 17' 5''$
Moon's node	202°20'0"	$\theta_{0\,\text{Moon' node}} = \left(\dfrac{A \times 232{,}300}{1{,}577{,}917{,}500}\right)f \times 360^\circ + 202^\circ 20' 0''$

For the two eclipses that we are considering, the longitudes of the sun and the moon are listed at $5^h 29^m$ Indian Standard Time (IST) which is the local time at the present standard meridian of India, whose terrestrial longitude is 82.5°, whereas our mean longitudes are at $6^h 0^m$ local time at Ujjain whose terrestrial longitude is 75.78°. So, we have to do two corrections to our mean longitudes to be able to compare with the tabulated values at $5^h 29^m$ for a terrestrial longitude of 82.5°.

The mean longitude of the planet at $5^h 29^m$ local time at Ujjain is given by:

$$\theta_{0,1} = \theta_0 - \left(\text{Mean rate of motion} \times \frac{31}{24 \times 60}\right).$$

The mean longitude at $5^h 29^m$ local time at the Indian standard meridian, that is, at the Indian Standard Time (IST) is given by:

$$\theta_{0,2} = \theta_{0,1} - \left(\text{Mean rate of motion} \times \frac{(82.5 - 75.78)}{24 \times 15}\right).$$

TRUE LONGITUDES OF THE SUN AND THE MOON

In the Indian astronomical tradition, at least from the time of Āryabhaṭa (499 CE), the procedure for calculating the geocentric longitudes of the sun and the moon consists essentially of two steps: first, the computation of the mean longitude of the planet known as the *madhyama graha* and second, the computation of the true or observed longitude of the planet known as the *sphuṭa graha*. The mean longitude is calculated for the desired day by computing the number of mean civil days elapsed since the epoch (this number is called the *ahargaṇa*) and multiplying it by the mean daily motion of the planet, and adding any epochal correction. Having obtained the mean longitude, a correction known as *manda-saṁskāra* (*manda*-correction) is applied to it. In essence, this correction takes care of the eccentricity of the planetary orbit due to its elliptical nature. The equivalent of this correction is termed the "equation of centre" in modern astronomy, and is a consequence of the elliptical nature of the orbit. The longitude of the planet obtained by applying the *manda*-correction is known as the *manda sphuṭa graha*, or simply the *manda sphuṭa*. The *manda*-correction is the only correction that needs to be applied in case of the sun and the moon for obtaining their true longitudes (*sphuṭa graha*). So, the *manda sphuṭa* is the true longitude in their case. We will now briefly discuss the details of this correction using the "epicyclic" or "eccentric" models.

In *fig*. 21.1, O is the centre of the *kakṣyā maṇḍala* (deferent) on which the mean planet P_0 is moving with a mean uniform velocity. $O\Gamma$ is the reference line which is in the direction of Meṣādi (first ponit of Aries). The deferent is taken to be of radius R, known as the *trijyā* which is the radius of a circle whose circumference is 21,600 units which is the number of minutes in 360°. The value of R is nearly 3,438. Around the mean planet P_0, a circle of radius r is to be drawn. This circle is known as the *manda-nīcocca-vṛtta*, or simply as *manda-vṛtta* (*manda*-epicycle). The texts specify the value of the radius of this circle r ($r \ll R$), in appropriate measure, for each planet. At any given instant of time, the true planet P is to be located on this epicycle by drawing a line from P_0 along the direction of the *mandocca*, or the apogee (parallel to OU). The point of intersection of this line with the epicycle gives the location of the planet P. The longitude of the mean planet P_0 moving on this circle is given by

$$\Gamma \hat{O} P_0 = \text{Mean longitude} = \theta_0.$$

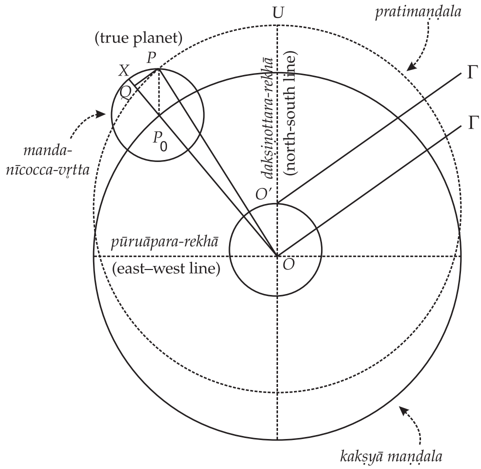


fig. 21.1: The Epicyclic and Eccentric Models of Planetary Motion

The true longitude of the planet is given by $\Gamma\hat{O}P_0$ which is to be obtained from θ_0. This is known as the "epicycle" model.

Alternatively, one could draw a *manda*-epicycle of radius r centred around O, which intersects OU at O'. With O' as the centre, a circle of radius R (shown by dashed lines in the figure) is drawn. This is known as the *pratimaṇḍala*, (eccentric circle). Since P_0P and OO' are equal to r and they are parallel to each other, $O'P = OP_0 = R$. Hence, P always lies on a circle of radius R, which is known as the eccentric circle. Also,

$$\Gamma\hat{O}'P = \Gamma\hat{O}'P_0 = \text{Mean longitude} = \theta_0.$$

Thus, the true planet P can be located on an eccentric circle of radius R centred at O' (which is located at a distance r from O in the direction of the apogee), simply by marking a point P on it such that $\Gamma\hat{O}'P$ corresponds to the the mean longitude of the planet. Since this process involves only an eccentric circle, without making a reference to the epicycle, it is known as the eccentric model. Clearly, the two models are equivalent to each other.

The procedure for obtaining the true longitude by either of the two models involves the longitude of the *mandocca* (apogee). In *fig*. 21.1, OU represents the direction of the *mandocca*, whose longitude is given by

$$\Gamma\hat{O}U = \textit{mandocca} \text{ (apogee)} = \theta_m$$

θ represents the true longitude which is to be determined from the position of the mean planet P_0. Clearly,

$$\theta = \Gamma\hat{O}P = \Gamma\hat{O}P_0 - P\hat{O}P_0 = \theta_0 - \Delta\theta.$$

Here, $\Delta\theta = P\hat{O}P_0$ is the correction-term. Since the mean longitude of the planet, θ_0 is known, the true longitude, θ is obtained by simply subtracting $\Delta\theta$ from θ_0. The expression for $\Delta\theta$ can be obtained by making the following geometrical construction. We extend the line OP_0, which is the line joining the centre of the *kakṣyā maṇḍala* and the mean planet, to meet the epicycle at X. From P drop the perpendicular PQ onto OX. Then:

$$U\hat{O}P_0 = \Gamma\hat{O}P_0 - \Gamma\hat{O}U = \theta_0 - \theta_m,$$

is the *manda-kendra* (anomaly) whose magnitude determines the magnitude of $\Delta\theta$. Also, since P_0P is parallel to OU (by construction), $P\hat{P}_0Q = \theta_0 - \theta_m$. Hence, $PQ = r\sin(\theta_0 - \theta_m)$, and $P_0Q = r\cos(\theta_0 - \theta_m)$. Since the triangle OPQ is right-angled at Q, the hypotenuse $OP = K$ (known as the *manda-karṇa*) is given by

$$K = OP = \sqrt{OQ^2 + QP^2} = \sqrt{\left(OP_0 + P_0Q\right)^2 + QP^2}$$

$$= \sqrt{\left\{R + r\cos\left(\theta_0 - \theta_m\right)\right\}^2 + r^2\sin^2\left(\theta_0 - \theta_m\right)}$$

Again from the triangle POQ, we have

$$K\sin\Delta\theta = PQ = r\sin(\theta_0 - \theta_m)$$

Hence,

$$\sin\Delta\theta = \sin(\theta_0 - \theta_m) = \frac{r}{K}\sin(\theta_0 - \theta_m)$$

Now in most of the Indian astronomy texts, is not a constant, but varies such that $\frac{r}{K}$ is a constant. $\frac{r}{K}$ is writen as $\frac{r_0}{R}$, where r_0 is the mean or tabulated value of the radius of the *manda* epicycle. Hence, the true longitude, θ is given by the expression:

$$\theta = \theta_0 - \sin^{-1}\left[\frac{r_0}{R}\sin\left(\theta_0 - \theta_m\right)\right]. \tag{1}$$

For the sun, $\frac{r_0}{R} = \frac{3}{80}$, and $\theta_m = 78°$. For the moon, $\frac{r_0}{R} = \frac{7}{80}$, and θ_m

increases at a constant rate.

Hence, the true longitudes of the sun and the moon are given by:

$$\theta_{\text{sun}} = \theta_{0\,\text{sun}} - \sin^{-1}\left[\frac{3}{80}\sin\left(\theta_{0\,\text{sun}} - 78°\right)\right]. \tag{2}$$

$$\theta_{\text{moon}} = \theta_{0\,\text{moon}} - \sin^{-1}\left[\frac{7}{80}\sin\left(\theta_{0\,\text{moon}} - \theta_{m\,\text{moon}}\right)\right]. \tag{3}$$

TRUE DAILY OTIONS OF THE SUN AND THE MOON

Verses 53-54 in "Sphuṭaprakaraṇam" (True Longitudes of Planets) in *Tantrasaṁgraha* give the expression for the "instantaneous" velocity of a planet, after discussing the *manda*-correction to a planet:

> Let the product of the *koṭiphala* (in miuutes) and the daily motion of the *kendra* be divided by the square root of the square of the *bāhuphala* of the moon subtracted from the square of *trijyā*. The quantity thus obtained has to be subtracted from the daily motion [of the moon] if [the *kendra* lies within the six signs] beginning from Makara and is to be added to the daily motion if [the *kendra* lies within the six signs] beginning from Karkaṭaka. This will be accurate (*sphuṭatarā*) value of the instantaneous velocity can be obtained (*tatsamayajāgati*) of the moon. For the sun also [the instantaneous velocity can be obtained similarly].

These verses clearly state that at any instant, the velocity or the true daily motion of the sun or the moon is given by:

$$\frac{d\theta}{dt} = \frac{d\theta_0}{dt} - \frac{r_0\cos(\theta_0 - \theta_m)\frac{d(\theta_0 - \theta_m)}{dt}}{\sqrt{R^2 - r_0^2\sin^2(\theta_0 - \theta_m)}}.$$

where θ_0 is the mean longitude, θ_m is the *mandocca* of the planet, $\theta_0 - \theta_m$ is the (*manda*) *kendra*, r_0 is the radius of the epicycle and R is the radius of the deferent. $r_0 \cos(\theta_0 - \theta_m)$ is the *koṭiphala*, and $r_0 \sin(\theta_0 - \theta_m)$ is the *bāhuphala*. The first term corresponds to the mean velocity and the second term corresponds to the *manda*-correction. This correction term can be written as:

$$-\frac{\frac{r_0}{R}\cos(\theta_0-\theta_m)\frac{d(\theta_0-\theta_m)}{dt}}{\sqrt{1-\frac{r_0^2}{R^2}\sin^2(\theta_0-\theta_m)}}.$$

The true daily motion of the planet can then be written as:

$$\frac{d\theta}{dt}=\frac{d\theta_0}{dt}-\frac{\frac{r_0}{R}\cos(\theta_0-\theta_m)\frac{d(\theta_0-\theta_m)}{dt}}{\sqrt{1-\frac{r_0^2}{R^2}\sin^2(\theta_0-\theta_m)}}.$$

It can be easily seen that this expression can be got by taking the derivative of the expression for the true longitude, θ in terms of the mean longitude, θ_0, the apogee, θ_m, and the epicycle radius, r_0. Here, it can be mentioned that the instantaneous velocity was first discussed by Bhāskara II in his celebrated work, *Siddhāntaśiromaṇi*, in 1150 CE itself. There, he had essentially used the approximation, $\sin^{-1}x \approx x$, for small x. In *Tantrasaṁgraha*, no such approximation is made and the correct expression for the derivative of the inverse sine function is used. This is in 1500 CE! This is truly amazing.

For the sun, $\frac{r_0}{R}=\frac{3}{80}$ and θ_m = 78°. For the moon, $\frac{r_0}{R}=\frac{7}{80}$ and θ_m is its apogee which increases constantly. Then, their true daily motions are given by the expressions:

$$\frac{d\theta_{\text{sun}}}{dt}=\frac{d\theta_{0\,\text{sun}}}{dt}-\frac{\frac{3}{80}\cos(\theta_{0\,\text{sun}}-78°)\frac{d(\theta_0-78°)}{dt}}{\sqrt{1-\frac{3^2}{80^2}\sin^2(\theta_0-78°)}}, \tag{4}$$

$$\frac{d\theta_{\text{moon}}}{dt}=\frac{d\theta_{0\,\text{moon}}}{dt}-\frac{\frac{7}{80}\cos(\theta_{0\,\text{moon}}-\theta_{m\,\text{moon}})\frac{d(\theta_{m\,\text{moon}}-\theta_{m\,\text{moon}})}{dt}}{\sqrt{1-\frac{7^2}{80^2}\sin^2(\theta_{0\,\text{moon}}-\theta_{m\,\text{moon}})}}. \tag{5}$$

TIME OF CONJUNCTION OF THE MOON AND THE EARTH'S SHADOW

Possibility of a Lunar Eclipse

The earth's shadow always moves along the ecliptic and its longitude will be exactly 180° plus that of the longitude of the sun. When the moon is close to the shadow and both of them are near a node, then there is a possibility of a lunar eclipse. This means that the latitude of the moon should be small. This situation is depicted in *fig*. 21.2, where *C* represents the *chāyā* (shadow), and *A* and *B* are the positions of the moon before and after the lunar eclipse.

Computation of the Instant of Conjunction

Usually, the longitudes of the planets are calculated at sunrise on a particular day. Let θ_s, θ_m and θ_c be the true longitudes of the sun, the moon and the *chāyā* (earth's shadow) respectively. Then, obviously,

$$\theta_c = \theta_s - 180^{\circ}. \tag{6}$$

When the longitudes of the moon and the earth's shadow are the same, the sun will be exactly at 180° from the moon. Since the sun and the moon are diametrically opposite each other at this instant, they are said to be in opposition. In order to determine this instant, the true longitudes of the sun (θ_s) and the moon (θ_m), are first calculated at sunrise on a full moon day. Then, the difference in longitudes of the moon and the *chāyā*, given by

$$\Delta\theta = \theta_m - \theta_c \tag{7}$$

is computed. The sign of $\Delta\theta$ indicates if the instant of opposition is over or is yet to occur.

1. If $\Delta\theta < 0$, it means that the instant of opposition is yet to occur, as the moon moves eastward with respect to the sun.
2. If $\Delta\theta > 0$, it means that the instant of opposition is already over. The positions of the moon corresponding to these two situations are indicated by *A* and *B* in *fig*. 21.2. Let Δt be the time interval between sunrise and the instant of opposition in *ghaṭikā*s or in *nāḍikā*s. Note that there are 60 *ghaṭikā*s in a civil day. Then Δt is computed using the relation,

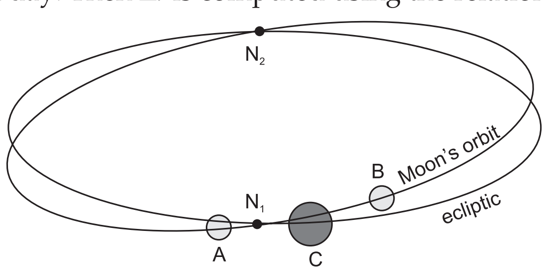


fig. 21.2: Possibility of a Lunar Eclipse.

$$\Delta t = \frac{|\Delta\theta|}{d_m - d_s} \times 60 \tag{8}$$

where d_m and d_s are the true daily motions of the sun ($\frac{d\theta_{\text{sun}}}{dt}$) and the moon ($\frac{d\theta_{\text{moon}}}{dt}$) which are given in the equations (4) and (5) respectively. The above expression for Δt (in *ghaṭikā*s) is obviously based upon the rule of three. If 60 *ghaṭikā*s correspond to a difference in longitude $d_m - d_s$, what is the time interval Δt, corresponding to the longitude difference, $|\Delta\theta|$?

Having determined Δt, the time of opposition of the sun and the moon, or equivalently the conjunction of the moon and the earth's shadow at the end of the full moon day, which is the same as the middle of the eclipse denoted by t_m, is obtained using the relation:

$$t_m = \text{Sunrise time} \pm \Delta t. \tag{9}$$

We have to use "+" if the instant of opposition is yet to occur and "−" otherwise.

EXACT MOMENT OF CONJUNCTION BY ITERATION

The instant of conjunction calculated using (9) is only approximate, as Δt used in the expression is found using a simple rule of three, that presumes uniform rates of motion for the sun and the moon, which is not true. In order to take the non-uniform motion into account, an iterative procedure to determine the true instant of conjunction is described here.

As per the computational scheme followed by Indian astronomers, the instant of sunrise or sunset is the reference point for finding the time of any event. Hence, the instant of true sunrise is first to be determined accurately. It was noted that this involves the application of the *cara* (ascensional difference), and the equation of time, where the latter has two parts, namely the correction due to the equation of centre and the correction due to the *prāṇakalāntara*. Here it is prescribed that the *cara* and the equation of time are to be determined at the instant of conjunction, in order to find the instant of true sunrise or sunset as the case may be. In this paper, we do not follow this procedure. We find the true longitudes of the sun and the moon directly at $5^h 29^m$ Indian Standared Time (IST) on the given day. All the times would refer to the IST. This is because, we can then directly compare our computations with the ephemeris values (*Rāṣṭrīya Pañcāṅga*).

First, the true longitudes of the sun and the moon are found at $5^h 29^m$ IST on the full moon day. Next, Δt is found using equation (8), and the first approximate value of the instant of conjunction, t_m is found using equation (9). The true longitudes of the sun and moon, and their true daily motions are determined at this instant using the procedure described in the previous sub-section, and $\Delta\theta$ is found at this instant. The second approximate value of the instant of conjunction is now determined using equations (8) and (9). The true longitudes and the daily motions are again computed at this instant, and the third approximate value is found using equations (8) and (9). This iteration process is carried on till two successive values of the instant of conjunction are the same to the desired accuracy.

MOON'S LATITUDE

The expression for the latitude, β of the moon is given by:

$$\sin\beta = \sin i \sin(\theta_m - \theta_n),$$

where i is the inclination of the moon's orbit, and θ_m, θ_n are the true longitudes of the moon and its ascending node respectively. When i is small,

$$\beta \approx i \sin(\theta_m - \theta_n).$$

In Indian astronomy texts, i is taken to be $i = 4.5^0 = 270'$. Then the formula given for the latitude β of the moon is,

$$\beta = \frac{270' \times R\sin(\theta_m - \theta_n)}{R} \tag{10}$$

where R is the *trijyā*, whose value is taken to be 3,438 minutes, and 270' is the inclination of the moon's orbit in minutes. The latitude thus obtained is in minutes which should be less than that of sum of the semi-diameters of the shadow and the moon, for a lunar eclipse to occur.

THE TIME OF HALF-DURATION, THE FIRST AND THE LAST CONTACT

The expression for the half-duration of the eclipse and the procedure to determine the instants of the beginning and the end of the eclipse may be understood with the help of *fig*. 21.3. Here O represents the centre of the shadow, and X is the centre of the moon's disc as it is about to enter into the shadow.

The total duration of the eclipse is made up of two parts:

1. The time interval, Δt_1, between the *sparśa*, which is the instant at which the moon enters the shadow and the instant of opposition (t_m).
2. The time interval, Δt_2 between the instant of opposition (t_m), and the *mokṣa*, which is the instant of complete release.

The suffixes 1 and 2 refer to the first and the second half-durations of the eclipse respectively. Though one may think naively that these two durations must be equal, this is not so because of the continuous change in the longitude of the sun, the moon and moon's nodes. Let r_1 and r_2 be the radii of the discs of the earth's shadow in the path of the moon and moon itself. In *fig*. 21.3, AX and OX represent the latitude (β) of the moon and the sum of the radii of the shadow and the moon, that is, $r_1 + r_2$, respectively. If d_m and d_s refer to the true rates of motions of the moon and the sun at the middle of the eclipse, the first half-duration of the eclipse in *ghaṭikā*s or *nāḍikā*s is found using the relation:

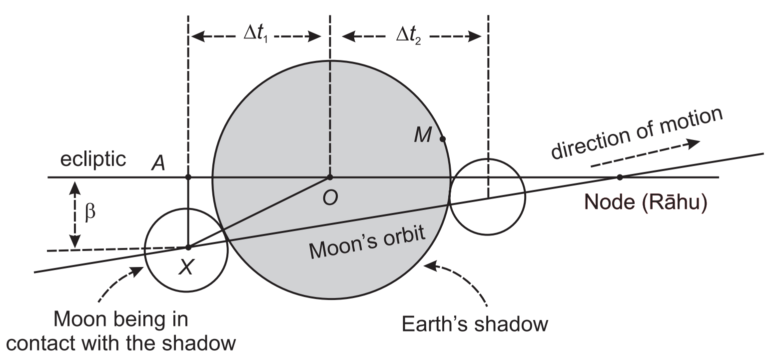


fig. 21.3 : First and the Second Half-durations of a Lunar Eclipse

$$\Delta t_1(\text{in } ghaṭikā\text{s}) = \frac{OA \times 60}{\text{diff. in daily motion}} = \frac{\sqrt{OX^2 - AX^2}}{d_m - d_s} \times 60$$

$$= \frac{\sqrt{(r_1 + r_2)^2 - \beta^2}}{d_m - s_s} \times 60 \tag{11}$$

Here the factor 60 represents the number of *ghaṭikā*s or *nāḍikā*s in a day. In the above expression, β is the latitude of the moon at the *sparśa* or the beginning of the eclipse. However, the instant of the beginning of the eclipse is yet to be determined, and hence the latitude of the moon at the beginning is not known. Moreover, the latitude of the moon is a continuously varying quantity. What is prescribed in the text *Tantrasaṁgraha* is an iterative procedure for finding the half-duration. As a first approximation, the latitude known at the instant of opposition is taken to be β and Δt_1 is determined. The iterative procedure to be adopted is described in the following section.

HALF-DURATIONS: ITERATION METHOD

The positions of the sun and the moon at the time of contact are now by subtracting their motions during the first-half duration from their values at the instant of opposition. The motion of the sun/moon is obtained by multiplying their true daily motions (d_m, d_s) by the half duration – the first approximation of which has been found as given by (11) – and dividing by 60 (the number of *nāḍikā*s in a day). This is done for the node also (but applied in reverse, as its motion is retrograde), whose longitude is required for the computation of moon's latitude. The latitude of the moon, β is now calculated using the values of d_m and d_s at the first contact. Δt_1 is now calculated using this value of β. This is the second approximation to it. The iteration procedure is carried on till the successive approximations to the half-durations are not different from each other to a desired level of accuracy. The procedure is the same for computing the second half-duration (*mokṣa kāla*), except that the positions of the sun and moon at the time of the *mokṣa* (release) are obtained by adding their motions during the second half-duration to their values at the instant of opposition.

FIRST AND SECOND HALF-DURATIONS OF TOTALITY

The beginning and end of totality are depicted in *fig* 21.4. Totality is the moment when the earth's shadow covers the moon fully. The procedure to find the two half-durations of totality is discussed below. The procedure is the same as the one for first and second half-durations of the eclipse, as a whole, with $r_1 + r_2$ replaced by $r_1 - r_2$ in the relevant expressions. Let $T(1)$ (in minutes) be the first half duration of totality.

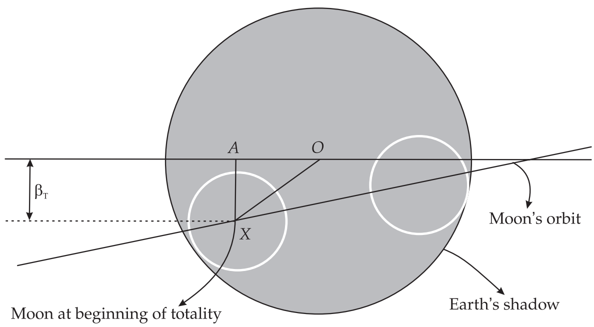


fig. 21.4: The First and Second Half-durations of Totality of a Lunar Eclipse

The expression for $T(1)$ is given by,

$$T(1) = \frac{\sqrt{(r_1 - r_2)^2 - \beta^2}}{d_m - d_s} \times 60,$$

where β is the latitude of the moon at the beginning of totality. However, the instant of the beginning of the totality is yet to be determined and hence the latitude of the moon at the beginning of totality is not known. As a first approximation, β is taken to be the latitude known at the instant of opposition, and $T(1)$ is determined. Then, an iteration procedure which is the same as the one for the first half-duration of the eclipse as a whole, is used to find the first half-duration of totality to the desired accuracy. The second half-duration of totality is determined in the same manner.

ANGULAR RADII OF THE EARTH'S SHADOW AND THE MOON'S DISC

The average radii of the orbits of the sun and moon (r_s and r_m), and the actual linear radii of the sun, the moon and the earth (R_s, R_m and R_e are given in the Table 21.4).

Table 21.4: The Radii of the Sun, Moon and Their Orbits, and the Radius of the Earth

Radii of	Notation used	Radius in Yojanas
Orbit of the sun	r_s	459,620
Orbit of the moon	r_m	34,380
Sun	R_s	2,205
Moon	R_m	157.5
Earth	R_e	525.21

From *fig* 21.5, it is clear that the angular radius, α of the moon is $\frac{R_m}{r_m}$. This is in radians. We have to multiply this by 3,438, which is the number of minutes in a radian, to obtain the value in minutes. Hence, the angular radius of the moon in minutes is given by

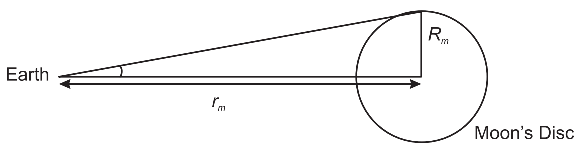


fig 21.5: Distance between Earth and Moon's Disc

$$r_2 = \frac{R_m}{r_m} \times 3,438 = \frac{157.5}{34,380} \times 3,438 = 15.75' \cdot$$

Angular Radius of Earth's Shadow

In *fig* 21.6:

$$\text{Radius of the sun} = AS - R_s,$$

$$\text{Radius of the earth} = CE - R_e,$$

$$\text{Radius of the earth's shadow} = GF - R_{sh}.$$

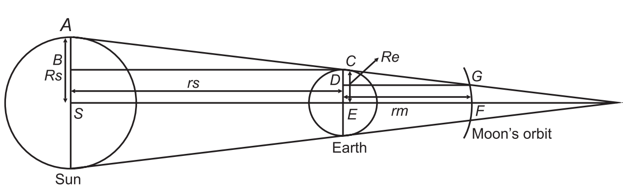


fig 21.6: Angular Radius of the Earth's Shadow
(not to scale)

It is clear that the triangles *ABC* and *CDG* are similar. Hence,

$$\frac{CD}{DG} = \frac{AB}{BC}.$$

Now, $CD = CE - DE = CE - GF = R_e - R_{sh}$.

$DG = EF = r_m$ (Radius of the moon's orbit).

$AB = AS - BS = AS - CE = R_{sh} - R_e$.

$BD = SE = r_s$ (Radius of the sun's orbit) .

Hence, $\dfrac{R_e - R_{sh}}{r_m} = \dfrac{R_{sh} - R_e}{r_s}$, or

$$\frac{R_{sh}}{r_m} = \frac{R_e}{r_m} - \frac{R_{sh} - R_e}{r_s}$$

Now $\frac{R_{sh}}{r_m}$ is the angular radius of the earth's shadow in radians. Hence angular radius of the earth's shadow (at the moon) in minutes,

$$\begin{aligned} r_1 &= \left(\frac{R_e}{r_m} - \frac{R_{sh} - R_e}{r_s}\right) \times 3{,}438 \\ &= \frac{525.21}{34{,}380} - \frac{(2{,}205 - 525.21)}{459{,}620} \\ &= 39.96'. \end{aligned}$$

Here, it should be noted that r_1 and r_2 are the average values of the angular radii of the earth's shadow and the moon's disc. The actual radii vary with time as radii of the sun's and moon's orbit vary with time.

LUNAR ECLIPSE ON 27/28 JULY 2018: COMPUTATIONS BASED ON TANTRASAṀGRAHA

For the demonstration of the above procedure to find the instant of opposition and half-duration at the time of release and contact during lunar eclipse, let us consider the lunar eclipse which happened on 27/28 July 2018 which was a total lunar eclipse which lasted about 3^h55^m. The duration of totality was 1^h44^m. Now, the *ahargaṇa* for 22 March 2001 is known to be 1,863,525 (Ramasubramanian and Sriram 2011). Then the *ahargaṇa* for 27 July 2018 is easily calculated to be 1,869,861.

True Longitude of the Sun and Its Rate of Motion on 27 July 2018, 5^h 29^m IST.

The mean longitude of the sun at 6^h at Ujjain is given by,

$$\theta_0 = \frac{A \times 4{,}320{,}000}{1{,}577{,}917{,}500}$$

$$= 5119.278746 \text{ revolutions.}$$

The mean longitude of the sun in degrees can be obtained by taking only the fractional part of the above value and multiplying it by 360, that is,

$$= 0.278746 \times 360$$

$$= 100.348^\circ.$$

The mean longitude of the sun at 5^h29^m (in degrees) at Ujjain is given by,

$$\theta_{01\ \text{sun}} = \theta_0 - \text{Mean rate of motion} \times \frac{31}{24 \times 60}$$

$$= 100.348 - 0.985602859 \times 360$$

$$= 100.327^\circ$$

The mean longitude of the sun at Indian Standard Meridian (ISM) is given by:

$$\theta_{02\ \text{sun}} = \theta_{01\ \text{sun}} - 0.985602859 \times \frac{(82.5 - 75.78)}{24 \times 15}$$

Hence, the mean longitude of the sun at $5^h\,29^m$ IST in deg. min. and sec. is given by:

$$\theta_{02\ \text{sun}} = 100.309^\circ = 100^\circ18'32''$$

Using the formula for the true longitude of the sun in terms of the mean longitude, the true longitude of the sun is:

$$\theta_{\text{sun}} = 100^\circ18'32'' - \sin^{-1}\left(\frac{3}{80}\sin\left(100.309^\circ - 78^\circ\right)\right)$$

$$= 99.493^\circ = 99^\circ29'35''.$$

The true daily motion of the sun can be obtained as,

$$\dot{\theta}_{\text{Sun}} = 0.985602859 - \frac{\frac{3}{80}\cos(100^\circ18'32'' - 78^\circ)}{\sqrt{1 - \frac{3^2}{80^2}\sin^2(100^\circ18'32'' - 78^\circ)}} \times 0.985602859$$

$$= 0^\circ57'5''$$

True Longitude of the Moon and Its Rates of Motion on 27 July 2018, $5^h\ 29^m$ IST

Including the *dhruva* of the moon, the mean longitude of the moon 6^{h} at Ujjain is given by

$$\theta_0 = \frac{A \times 57{,}753{,}320}{1{,}577{,}917{,}500} \text{ revln.} + 4.7627777^{\circ}$$

$$= 68438.736935562221 \text{ revln.} + 4.7627777^{\circ}$$

$$= 0.736935562221 \times 360^{\circ} + 4.7627777^{\circ}$$

$$= 270.060^{\circ}$$

The mean longitude of the moon at $5^{\mathrm{h}}29^{\mathrm{m}}$ local time at Ujjain is given by,

$$\theta_{01 \text{ Moon}} = \theta_{01} - \text{Mean rate of motion} \times \frac{31}{24 \times 60}$$

$$= 270.060^{\circ} - 13.17635124 \times \frac{31}{24 \times 60}$$

$$= 269.\ 776^{\circ}.$$

The mean longitude of the moon at $5^{\mathrm{h}}29^{\mathrm{m}}$ IST is given by,

$$\theta_{02 \text{ moon}} = \theta_{01 \text{ moon}} - 13.17635124 \times \frac{(82.5 - 75.78)}{24 \times 15}$$

$$= 269.530^{\circ}.$$

Hence, the mean longitude of the moon at $5^{\mathrm{h}}29^{\mathrm{m}}$ IST in deg. min. and sec. is given by,

$$\theta_{0 \text{ moon}} = 269.530^{\circ} = 269^{\circ}31'48''$$

The longitude of the moon's apogee at 6^{h} at Ujjain including the *dhruva* is given by,

$$\theta_{0 \text{ moon's apogee}} = \left\{\frac{A \times 488{,}122}{1{,}577{,}917{,}500}\right\} \times 360^{\circ} + 119.28472^{\circ}$$

$$= 0.4334675581 \times 360^{\circ} + 119.28472^{\circ}$$

$$= 275.330^{\circ}$$

The mean longitude of the moon's apogee at $5^{\mathrm{h}}29^{\mathrm{m}}$ local time at Ujjain is given by,

$$\theta_{01\,m} = \theta_0 - \text{Mean rate of motion} \times \frac{31}{24 \times 60}$$

$$= 275.333 - 0.111364453 \times \frac{31}{24 \times 60}$$

$$= 275.331^{\circ}.$$

The mean longitude of the moon's apogee at 5^h29^m IST is given by

$$\theta_{02\,m} = \theta_{01\,m} - 0.111364453 \times \frac{(82.5 - 75.78)}{24 \times 15}$$
$$= 275.329^o$$
$$= 275^o19'44''.$$

Using the formula for the true longitude of the moon in terms of its mean logitude and its apogee, the true longitude of the moon is

$$\theta_{moon} = 269°31'48'' - \sin^{-1}\left(\frac{3}{80}\sin\left(269°31'48'' - 275°19'44''\right)\right)$$
$$= 270.037° = 270°2'13''$$

The true daily motion of the moon can be obtained as,

$$\dot{\theta}_{moon} = 13.17635124° - \frac{\frac{7}{80}\cos(270°2'13'' - 275°19'44'')}{\sqrt{1 - \frac{7^2}{80^2}\sin^2(270°2'13'' - 275°19'44'')}}$$
$$\times (13.1763512 - .111364453) = 12°2'20''$$

Computation of the Instant of Opposition, or the Middle of the Eclipse, t_m.

The longitude of the earth's shadow is:

$$\theta_c = \theta_s + 180^o$$
$$= 99^o29'35''$$
$$= 279^o29'35''.$$

Then, the time interval between 5^h29^m IST, and the instant of opposition in hours is

$$\Delta t = \frac{\left|\theta_{sun} - \left(\theta_{moon} - 180°\right)\right|}{d_m - d_s} \times 24$$
$$= \frac{|99°29'35'' - 90°2'13''|}{\left(12°2'20'' - 0°57'5''\right)}$$
$$= 20.468\,\text{hours.}$$

In this case, as the moon lags behind the shadow, we have:

Instant of opposition, $t_m = \Delta t$ + Initial time ($5^h\,29^m$)

$$= 20.468 + 5.483$$

$$= 25.951 \text{ hours} = 25^{h}\,57^{m}.$$

The instant of opposition obtained is by interpolation. We have to compute the longitudes of the sun and the moon at this intsant (t_m) and check whether they are actually in opposition. If they are not, an iteration method would have to be adopted to compute the true instant of opposition.

The mean longitude of the sun at t_m is.

$$\theta_{0\,\text{sun}}(t_m) = 100°18'32'' + 0.985602859 \times \frac{20.468}{24}$$

$$= 101.150° = 101°9'.$$

Therefore the true longitude of the sun at t_m is:

$$\theta_{\text{sun}}(t_m) = 101°9' - \sin^{-1}\left(\frac{3}{80}\sin\,(101°9' - 78°)\right).$$

$$= 100°18'18''.$$

The mean longitude of the moon at t_m is:

$$\theta_{0\,\text{moon}}(t_m) = 269°31'48'' + 13.17635129 \times \frac{20.468}{24}$$

$$= 280.767° = 280°46'2''.$$

The longitude of the moon's apogee at t_m is:

$$\theta_{\text{moon apogee}}(t_m) = 275°19'44'' + 0.111364453 \times \frac{20.468}{24}$$

$$= 275.424° = 275°25'26''.$$

Therefore, the true longitude of the moon at t_m is.

$$\theta_{\text{mon}}(t_m) = 280°46'2'' - \sin^{-1}\left(\frac{7}{80}\sin\,(280°46'2'' - 270°25'26'')\right).$$

$$= 280°18'0''.$$

Hence, we see that $\theta_{\text{sun}}(t_m) + 180° = 280°18'18''$, which is very close to $\theta_{\text{moon}}(t_m) = 280°18'0''$ already, and we stop here.

The daily Motions of the Sun and the Moon at the Middle of the Eclipse, t_m

The daily motion of the sun at *tm* can be obtained as:

$$\dot{\theta}_{sun} = 0.985602859 - \frac{\frac{3}{80}\cos(101°9' - 78°)}{\sqrt{1 - \frac{3^2}{80^2}\sin^2(101°9' - 78°)}}$$

$$\times 0.985602859 = 0°57'6''$$

The daily motion of the moon at t_m can be obtained as:

$$\dot{\theta}_{sun} = 13.17635124 - \frac{\frac{3}{80}\cos(280°46'2'' - 275°25'26'')}{\sqrt{1 - \frac{3^2}{80^2}\sin^2(280°46'2'' - 275°25'26'')}}$$

$$\times (13.17635124 - 111364453) = 12°2'17''.$$

Latitude of the Moon at the Instant of Opposition, or the Middle of the Eclipse, t_m

Now to determine the first half duration, we need to know the latitude of the moon at the instant of opposition which involves the longitude of the moon's node. Including the *dhruva* of the node, the longitude of the moon's node at Ujjain at 6 a.m. local time corresponding to the *ahargaṇa*, 1,869,861 is:

$$\theta_{\text{moon's node}} = -\frac{A \times 232,300}{1,577,917,500} + 202°20'$$

$$= -275.279734396 \text{ revln.} + 202°20'$$

$$= -101.704° + 202°20'$$

$$= 101.629°$$

Hence, the longitude at $5^h 29^m$ local time at Ujjain is

$$= 101.629° + 0.052998968 \times \frac{31}{24 \times 60}$$

$$= 101.630°.$$

Hence, the longitude of the moon's node at $5^h 29^m$ IST is:

$$= 101.630° + 0.052998968 \times \frac{(82.5 - 75.78)}{24 \times 15}$$

$$= 101.631°.$$

Hence, the longitude of the moon's node at the middle of the eclipse, t_m is

$$\theta_n = 101.631° - 0.052998968 \times \frac{20.468}{24}$$

$$= 101.586°$$

$$= 101°35'10''.$$

Then, the latitude of the moon at the instant of opposition is found to be:

$$\beta\,(t_m) = 270'\,\sin\,(280°18'0'' - 101°35'10'')$$
$$= 6.060'.$$

First Half-duration of the Lunar Eclipse

The sum of the radii of the earth's shadow-disk (r_1 = 39.96'), and the moon's disk (r_2 = 15.75'), $r_1 + r_2$ = 55.71'. Hence, the first approximation to the first haf-duration of the eclipse is given by:

$$\Delta t_{1/2}\,(11) = \frac{\sqrt{55.71^2 - 6.060^2}}{(722.28 - 57.096)} \times 24$$
$$= 1.998 \text{ hrs.} = 119.88 \text{ min.}$$

To obtain a more accurate value, an iteration procedure is involved. To find the second approximation to the first half duration, the longitudes of the moon and its node are found at the first approximation to the beginning of the eclipse. The longitude of the moon at this instant is:

$$\theta_m\,(\Delta t_{1/2}\,(11)) = 280.3° - \frac{1.998}{24} \times 12.038$$
$$= 279.298°$$

Longitude of the moon's node:

$$\theta_n\,(\Delta t_{1/2}\,(11)) = 101.586 + \frac{1.998}{24} \times 0.052998968$$
$$= 101.590°$$

Latitude of the moon:

$$\beta = 270'\,\sin\,(279.298 - 101.590)$$
$$= 10.80'.$$

Then, the second approximation to the first half duration is given by:

$$\Delta t_{1/2}\,(12) = \frac{\sqrt{55.71^2 - 10.8^2}}{(722.28 - 57.096)} \times 24$$
$$= 1.972 \text{ hrs.} = 118.32 \text{ min.}$$

The successive values of the first half-duration obtained are quite close, so we stop here. Therefore the first half duration of the lunar eclipse which happened on 27/28 July 2018 is 118.32 min., i.e. 1^h58^m.

First Half-duration of Totality

The difference of the radii of the earth's shadow-disk (r_1 = 39.96'), and the moon's disk (r_2 = 15.75'), $r_1 - r_2$ = 24.21'. Hence, the first approximation to the first haf-duration of totality of the eclipse, T (1), is given by:

$$T\,(11) = \frac{\sqrt{24.21^2 - 6.060^2}}{(722.28 - 57.096)} \times 24$$
$$= 0.846 \text{ hrs.} = 50.76 \text{ min.}$$

To find the second approximation to the first half-duration of totality, the longitudes of the moon and its node are found at the first approximation to the beginning of totality. The longitude of the moon at this instant is:

$$\theta_m(\mathrm{T}(1, 1)) = 280.300 - \frac{0.846}{24} \times 12.038$$
$$= 279.876^{\circ}$$

The longitude of the moon's node at this instant is:

$$\theta_n(\mathrm{T}(1, 1)) = 101.586 + \frac{0.846}{24} \times 0.052998968$$
$$= 101.588^{\circ}$$

Latitude of the moon at this instant is:

$$\beta = 270' \sin (279.876 - 101.588)$$
$$= 08.066'.$$

Then, the second approximation to the first half-duration is given by:

$$T(12) = \frac{\sqrt{24.21^2 - 8.066^2}}{(722.28 - 57.096)} \times 24$$
$$= 0.824 \text{ hrs.} = 49.44 \text{ min.}$$

The successive values of the first half-duration of totality obtained are quite close, so we stop here. Therefore the first half-duration of totality of the lunar eclipse, T (1) which happened on 27/28 July 2018 is 0.824 hrs., i.e. 49.44 min.

Second Half-duration of the Eclipse

The first approximation to the second half-duration as whole is the same as the first approximation to the first half-duration, as

β is taken at the middle of the eclipse, and therefore $\Delta t_{1/2}$ (21) = $\Delta t_{1/2}$ (11) = 1.998 hrs.

To find the second approximation to the second half-duration, the longitudes of the moon and its node are found at the first approximation to the end of the eclipse. The longitude of the moon at this instant is:

$$\theta_m (\Delta t_{1/2} (21)) = 280.300 + \frac{1.998}{24} \times 12.038$$
$$= 281.302^\circ.$$

The longitude of the moon's node at this instant is:

$$\theta_n (\Delta t_{1/2} (21)) = 101.586 - \frac{1.998}{24} \times 0.052998968$$
$$= 101.582^\circ.$$

Latitude of the moon:

$$\beta = 270' \sin (281.302 - 101.582)$$
$$= 1.319'.$$

Then, the second approximation to the first half-duration is given by:

$$\Delta t_{1/2} (22) = \frac{\sqrt{55.71^2 - 1.319^2}}{(722.28 - 57.096)} \times 24$$
$$= 2.009 \text{ hrs.} = 120.54 \text{ min.}$$

To find the third approximation to the second half-duration, the longitudes of the moon and its node are found at the second approximation to the end of the eclipse. The longitude of the moon at this instant is:

$$\theta_m (\Delta t_{1/2} (22)) = 280.300 + \frac{2.009}{24} \times 12.038$$
$$= 281.308^\circ.$$

The longitude of the moon's node at this instant is:

$$\theta_n (\Delta t_{1/2} (22)) = 101.586 - \frac{2.009}{24} \times 0.052998968$$
$$= 101.582^\circ.$$

Latitude of the moon:

$$\beta = 270' \sin (281.302 - 101.582)$$
$$= 1.291'.$$

Then, the third approximation to the second half-duration is given by:

$$\Delta t_{1/2}(23) = \frac{\sqrt{55.71^2 - 1.291^2}}{(722.28 - 57.096)} \times 24$$
$$= 2.009 \text{ hrs.} = 120.54 \text{ min.}$$

The successive values of the first half duration obtained are the same (to an accuracy of 0.001 hr.) . Hence, we stop here. Therefore the second half-duration of the lunar eclipse as a whole which happened on 27/28 July 2018 is 120.54 min, i.e. 2^h1^m.

Second Half-duration of Totality

The first approximation to the second half-duration of totality, $T(2)$ is the same as the first approximation to the first half-duration of totality, as β is taken at the middle of the eclipse, and therefore $T(21) = T(11) = 50.76$ min.

To find the second approximation to the second half-duration of totality, the longitudes of the moon and its node are found at the first approximation to the end of totality. The longitude of the moon at this instant is:

$$\theta_m\,(\mathrm{T}(2,1)) = 280.300 + \frac{0.846}{24} \times 12.038$$
$$= 280.724^\circ.$$

The longitude of the moon's node at this instant is

$$\theta_n\,(\mathrm{T}(2,1)) = 101.586 - \frac{0.846}{24} \times 0.052998968$$
$$= 101.584^\circ.$$

Latitude of the moon at this instant is

$$\beta = 270' \sin(280.724 - 101.584)$$
$$= 4.053'.$$

Then, the second approximation to the second half-duration of totality is given by:

$$T(22) = \frac{\sqrt{24.21^2 - 4.053^2}}{(722.28 - 57.096)} \times 24$$
$$= 0.861 \text{ hrs.} = 51.66 \text{ min.}$$

The successive computed values of the second half-duration of totality differ from each other by only 0.9 min, so we stop here. Therefore the second half-duration of totality, T (2) is 51.66 min.

COMPARISON BETWEEN THE COMPUTED VALUES AND THE VALUES OBTAINED IN "RĀṢṬRĪYA PAÑCĀṄGA" FOR SOME LUNAR ECLIPSES

In Table 21.5, we compare the various parameters tabulated in the condensed Indian ephemeris (*Rāṣṭrīya-Pañcāṅga*, 2017), and the values computed from the parameters and the procedure of *Tantrasaṁgraha*, as above. It is very remarkable that the two sets of values are close to each other.

A similar exercise was carried out for the lunar eclipse which occurred on 7 August 2017. For this, the *Tantrasaṁgraha* values and the modern values (Rashtriya Panchang, 2016) are compared in Table 21.6. Again there is a very remarkable agreement between the two sets. This eclipse was partial according to both the computed values and the values tabulated in the *Rāṣṭrīya-Pañcāṅga*.

Table 21.5: Comparison of the *Rāṣṭrīya-Pañcāṅga* Parameters and those Computed from the Procedure described in Tantrasaṁgraha for the Total Lunar Eclipse on 27/28 July 2018

	Rāṣṭrīya Pañcāṅga Value	*Calculated value from Tantrasaṁgraha*
θ_{sun} (Long. of the sun at 5^h29^m IST on 27 July 2018)	99°49'30"	99°29'35"
θ_{moon} (Long. of the moon at 5^h29^m IST on 27 July 2018)	270°37'44"	270°2'13"
t_m (Middle of the eclipse, w.r.t. the 0^h IST of 27 July 2018)	25^h52^m	25^h57^m
$\Delta t_{1/2}$ (1) (First half-duration of the eclipse)	118 min	118.32 min
$\Delta t_{1/2}$ (2) (Second half-duration of the eclipse)	117 min	120.54 min
T_1 (First half-duration of totality)	52 min	49.44 min
T_2 (Second half-duration of totality)	52 min	51.66 min

Table 21.6: Comparison of the Rāṣṭrīya-Pañcāṅga Parameters and those Computed from the Procedure described in Tantrasaṁgraha for the Partial Lunar Eclipse on 7 August 2017

	Rāṣṭrīya Pañcāṅga	*Calculated value*
θ_{sun} (at 5^h29^m IST)	110°35'34"	110°13'20"
θ_{moon} (at 5^h29^m IST)	281°54'23"	281°17'16"
t_m (Middle of eclipse)	23^h51^m	24^h
$\Delta t_{1/2}$ (1) (First half-duration)	59^m	1^m
$\Delta t_{1/2}$ (2) (Second half-duration)	58^m	1^m

Concluding Remarks

Indian astronomy texts are noted for their simplified calculational procedures for various kinds of variables in general and eclipses in particular. *Tantrasaṁgraha* of Nīlakaṇṭha Somayājī composed in 1500 CE is one of the major astronomy texts of the Kerala school, noted for many advancements including a major modification of the Indian planetary model. It had also been noticed that the longitudes of the sun and the moon computed from this work are fairly accurate (within a degree) even for recent dates. Hence it is worhwhile to check whether the eclipse calcuations using the *Tantrasaṁgraha* procedure and parameters are accurate. The solar eclipse calculations are very involved, as parallax plays an important role in them. Hence, we have confined ourselves to lunar eclipse computations only in this paper. We have given all the details of the procedure and illustrated it with the explicit example of the total lunar eclipse of 27/28 July 2018. We calculated the instant of opposition and the two haf-durations of the eclipse as a whole and also the two half-durations pertaining to the totality. We compared the computed values of these with the values tabulated in the Indian national ephemeris (Rāṣṭrīya-Pañcāṅga). The agreement is excellent. We performed the calcualtions for another eclipse (which was partial) on 7 August 2017. Again there is a remarkable agreement between the computed and the tabulated values. It would be worthwhile to carry out a detailed and systematic study of the accuracy of the *Tantrasaṁgraha* procedure and parameters for a large number of lunar and also solar eclipses with a statistical analysis, to establish its efficacy for eclipse-computations.

References

Ramasubramanian, K. and M. S., Sriram, 2011, *Tantrasaṁgraha of Nīlakaṇṭha Somayājī, with Translation and Mathematical notes*, New Delhi: Hindustan Book Agency (repr. Springer, 2011).

Rāṣṭrīya-Pañcāṅga, 2016, (English) for Śaka Era 1939, Kali Era 5117-18 (2017-18 CE), New Delhi: Positional Astronomy Centre, India Meteorological Department.

Rāṣṭrīya-Pañcāṅga, 2017, (English) for Śaka Era 1940, Kali Era 5118-19 (2018-19 CE), New Delhi: Positional Astronomy Centre, India Meteorological Department.

Shukla, K.S., K.V. Sharma, 1976, Āryabhaṭīya of Āryabhaṭa, Indian National Science Academy, New Delhi.

Sarma, K.V., 1977, *Tantrasaṁgraha of Nīlakaṇṭha Somayājī with the commentaries, Yuktidīpikā [for chapters I-IV] and Laghuvivṛti [for chapters V-VIII] of Śaṅkara Vārier*, Hoshiarpur: Vishveshvaranand Vishwabandhu Research Institute.

Sen, S.N. and K.S. Shukla (eds), 1985, *A History of Indian Astronomy*, New Delhi: Indian National Science Academy; revised edn, 2000.

Sriram M.S. and K., Ramasubramanian, 1994, *DST project report on "An Analysis of the Accuracy and Optimality of Algorithms in Indian Astronomy"*, Chennai: Department of Theoretical Physics, University of Madras.

22

Non-trivial Use of the "Trairāśika" (Proportionality Principle) in Indian Astronomy Texts

M.S. Sriram

Abstract: For the Indian astronomer-mathematicians, the rule of three, which is essentially the proportionality principle, and the theorem of the right triangle play a crucial role in the derivation of all the results related to the planetary positions and the diurnal problems. For instance, in his *Grahagaṇita* (planetary mathematics), a part of his magnum opus, *Siddhānta-Śiromaṇi* (*Crest-jewel of the Astronomical Treatises*), the celebrated Indian astronomer-mathematician Bhāskara (twelfth century CE) lists many latitudinal triangles (right triangles where one of the acute angles is the latitude of the place). Then very many relations which are of relevance to the shadow problems and the diurnal problems are derived using the similarity of triangles. These are straightforward applications of the proportionality principle. However, there are very non-trivial, far-from-direct applications of the proportionality principle also in Indian astronomy texts. In this article, three examples of these are considered. Two of them are considered by Bhāskara: one of them is the derivation of a second-order interpolation formula due to the great astronomer-mathematician Brahmagupta (seventh century CE), and the other is an expression for the part of the equation of time due to the obliquity of the ecliptic. The third is a relation involving the *vākya*s (mnemonics) for the longitude of the moon on 248 consecutive days.

Introduction

In explaining verse 246 of his *Līlāvatī*, Bhāskara remarks that, just as this universe is pervaded by Lord Nārāyaṇa in all his manifestations, "so is all this collections of instructions for computations pervaded by the rule of three terms". The rule of three is a very important topic in all Indian mathematical texts. This rule and its generalization to rules of five, seven, nine, etc. have wide applications in Vyavahāragaṇita (mathematics of (business and other) practices), and normally discussed in great detail with a large number of examples. It also has very many applications in astronomy. For instance, in the chapter on diurnal problems in the *Grahagaṇita* part of *Siddhānta-Śiromaṇi*, Bhāskarācārya lists eight important latitudinal triangles[1] in verses 13-17. As these are all similar triangles, the sides are in the same proportion in all of them. Then, we obtain relations among the various physical quantities of importance related to diurnal problems, like the zenith distance, hour angle, declination, latitude and azimuth. using the rule of three (repeatedly at times). In verse 29 of the *Siddhāṅta-Śiromaṇi* Bhāskara exclaims:

> There are 63 ways of obtaining the *pala jyā* ($R\sin\varphi$, where φ is the latitude), and the *lamba jyā* ($R\cos\varphi$). From the hundreds of ways of obtaining *agra jyā* (essentially, the distance between the rising–setting line and the east–west line), there are infinte ways of obtaining the *lamba jyā* and other quantities (using the rule of three).

Most of the applications of the rule of three in Indian mathematics and astronomy are somewhat direct and straightforward, as in the case of the latitudinal triangles mentioned above. However, there are some very non-trivial applications too, which lead to significant results. We discuss three such examples in this paper.

The Udayāntara Correction and the Proportionality Principle

In *fig*. 22.1, Γ is the first point of Aries, where the ecliptic intersects the equator, P is the pole of the equator and S is the sun. $\lambda = \Gamma S$

[1] Right triangles with the latitude of the location as one of the acute angles.

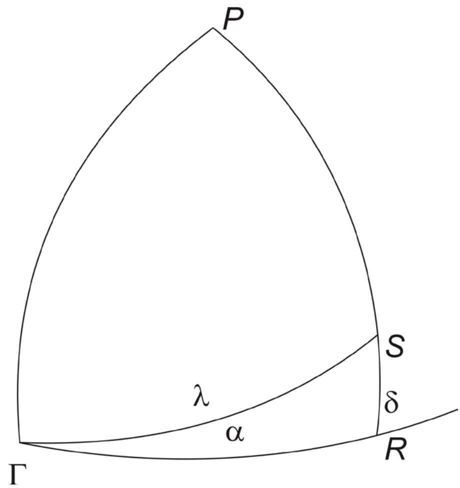


fig. 22.1: Longitude λ, Right Ascension α, and Declination δ.

is the *sāyana* (tropical) longitude of the sun, which is measured along the ecliptic, and α is the right ascension corresponding to λ, measured along the equator. This is the rising time of the ecliptic arc $\lambda = \Gamma S$ at the equatorial horizon. Let $\in$ be the obliquity of the ecliptic (the angle between the ecliptic and the equator) and δ be the declination of the sun. Then it can be shown that:

$$\sin\alpha = \sqrt{\frac{\sin^2\lambda - \sin^2\delta}{\cos\delta}} = \frac{\sin\lambda\cos\in}{\cos\delta},$$

where we have used $\sin\delta = \sin\in\,\sin\lambda$ for obtaining the second expression for $\sin\alpha$. The first expression was stated first in the Indian tradition by Śrīpati in his *Siddhāntaśekhara* (eleventh century CE), without any explanation. Bhāskara states both the expressions in his *Siddhānta-Śiromaṇi* and provides the rationale in the *upapatti* for the pertinent verses.

Now $\lambda - \alpha$ is the part of equation of time due to the obliquity of the ecliptic and is termed *udayāntara* in Indian astronomy. $\in = 24°$ in most Indian texts, which is also the maximum value of δ. Then $\lambda - \alpha$ is never too large. This is exploited by Bhāskara to give a simple expression for the *udayāntara*, $\lambda - \alpha$, based on *trairāśika* (rule of three):

$$\lambda - \alpha = 2.6° \sin(2\lambda).$$

This is how Bhāskara explains it in the *upapatti* (rationale) for the verse 65 of the *Siddhānta-Śiromaṇi*:

> *Upapatti* (Rationale): ... Find the *R*sine of the longitude (*R*sin λ, or the *dorjyā*) and the day-radius (*R* cos δ or the *dyujyā*) of the tropical mean sun. Divide the *dorjyā* by the *dyujyā* and multiply by *dyujyā* at the end of Mithuna (the third zodiacal sign, Gemini). The arc of the above in *asus*[2] subtracted from the mean tropical longitude of the sun in minutes is the true value of the *antara* in *asus*. By this is meant the *udayāntara*. In the middle of the quarter, this is slightly more than 26 *pala*s (or *vināḍī*s). To find it according to the *R*sine, the (mean longitude of the) sun is doubled. When the *R*sine of double the sun is found, then it (corresponding arc) becomes three signs (90°) at the middle of the quarter. Apply the rule of three for 26 and the *R*sine (of double the longitude). If for a *R*sine equal to *kharka* (120), we obtain a difference of 26, what is it for the desired *R*sine?

Here, Bhāskara gives a simple, approximate expression for the *udayāntara* correction, using an ingenious intuitive argument based on proportionality (Arkasomayaji 2000).

Now $\lambda - \alpha = 0$ when $\lambda = 0$ and $\lambda = 90°$. Hence $\lambda - \alpha$ cannot be proportional to sin λ[3]. Bhāskara argues that $\lambda - \alpha$ would be maximum at the middle of the quadrant, that is when $\lambda = 45°$. So, he proposes that $\lambda - \alpha$ is proportional to sin 2λ, or,

$$\lambda - \alpha = A \sin 2\lambda,$$

where A is a constant.

Now, $\sin \alpha = \dfrac{\sin\lambda \cos\epsilon}{\cos\delta}$. $\epsilon = 24°$ and $\sin\delta = \sin\epsilon \sin\lambda$. When

---

[2] Essentially, minutes; however, as it is a time unit, it is termed *asus*.

[3] A word of caution: In Indian texts, only *R*sines is used instead of the sine function and only arcs are considered instead of the angles. The *R*sine of an angle is the normal sine of the angle multiplied by the radius, *R* of the circle, which is normally taken to be $\frac{21600}{2\pi} \approx 3438$. However, a smaller value of *R* is also chosen at times. In the present context, Bhāskara uses a value of $R = 120$. In this section, we do not include *R* explicitly in the expressions which follow. Also, the unit of "degree" is used for the arcs in the Indian texts.

$\lambda = 45°$ we find that $\alpha = 42.41°$. Therefore, $\lambda - \alpha = 2.59° = A$, for $\lambda = 45°$. Also, $360° = 60$ *nāḍī* = 3600 *vināḍī*. Hence, $1° = 10$ *vināḍī*.

Hence, for a general λ,

$$\lambda - \alpha = 2.59° \sin(2\lambda) = 25.9 \textit{ vināḍī} \sin(2\lambda).$$

Bhāskara uses 2.6° instead of 2.59° and 26 instead of 25.9 *vināḍī*. Here, *udayāntara* is found directly and the expression for it is far simpler than what one would have got by computing α from the expression for $\sin \alpha$, finding the arc α from it, and subtracting it from λ.

Explanation of the Second-order Interpolation in Bhāskara's Siddhānta-Śiromaṇi

FIRST-ORDER INTERPOLATION FOR THE R SINES

The sine function plays an important role in most calculations in spherical astronomy. So, it is important to know the value of the sine function accurately for an arbitrary angle. Normally, in Indian astronomy texts, the quadrant is divided into 24 equal parts and the value of the *R*sine is specified for 24 angles which are integral multiples of $\frac{90°}{24} = 3°45'$. The value of the *R*sine of an intermediate angle is determined by linear interpolation which amounts to using the rule of proportions. This is how Bhāskara describes it in his *Siddhānta-Śiromaṇi* (Arkasomayahi 2000):

> *tattvāśvibhaktā asavaḥ kalā vā tallabdhasaṁkhyā gatasiñjinī sā* ||10||
> *yātaiṣyajīvāntaraśeṣaghātāt tattvāśvilabdhyā sahitepsitā syāt* |

> When the arc in minutes [corresponding to the desired *R*sine] is divided by 225, the obtained number (quotient) is the [number of] elapsed *R*sines. The remainder multiplied by the difference between the succeeding and preceding *R*sines and divided by 225, together [with the elapsed *R*sine] gives the desired *R*sine.

He explains it thus:

> *Upapatti* (rationale): Aren't the *R*sines 24 in number? In the circle, a quadrant consists of 5400 minutes. Each of the 24 divisions is equal to 225 minutes. Therefore, the elapsed minutes divided by 225 gives [number of] elapsed *R*sines. In the circle, the difference of *R*sines corresponds to an arc-difference of 225. Then if we

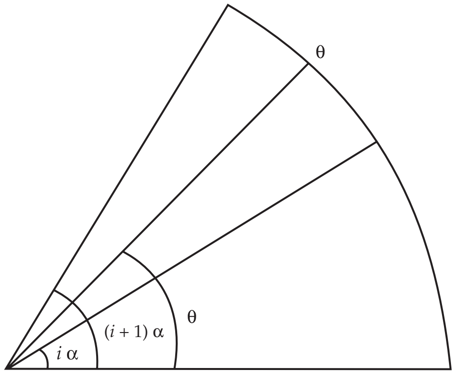


fig. 22.2: *R*sine of an intermediate angle

> obtain the (known) difference of *R*sines corresponding an arc-difference of 225, then what will it be for the remaining minutes? The result of this added to the previous *R*sine gives the desired result (desired *R*sine).

Bhāskara's *upapatti* can be understood thus:

The *R*sine table gives the values of the *R*sines for angles which are multiples of $\alpha = 225'$, i.e. $R \sin (i\alpha)$, $i = 1, \ldots, 24$. For an intermediate angle θ, the R sine is found from interpolation. Let θ be divided by α. Let the quotient be i and the remainder be Ψ, i.e. $\theta = i\alpha + \Psi$, $\Psi < 225'$. Then $R\sin \theta - R\sin i\alpha$ is found from the rule of proportions, that is, if the *R*sine difference is $R\sin [(i + 1)\alpha] - R\sin i\alpha$ for an angular difference α, then what is it for an angular difference Ψ? The answer is:

$$R\sin\theta - R\sin i\alpha = \frac{R\sin[(i+1)\alpha] - R\sin i\alpha}{\alpha} \times \Psi.$$

From this, $R\sin \theta$ is determined.

BRAHMAGUPTA'S SECOND-ORDER INTERPOLATION FORMULA EXPLAINED BY BHĀSKARA

Implicit in the linear interpolation formula is the assumption that the *R*sine varies uniformly within each of the twenty-four 225' intervals. This is reasonably accurate for most purposes. However, many texts, especially the *karaṇa* ones use larger angular intervals for simplicity in computational procedures, with shorter *R*sine tables. For instance, Brahmagupta uses an interval of 15° (that is a sine table with only six entries) in his celebrated *karaṇa* text, *Khaṇḍakhādyaka* (Sengupta 1934). Bhāskara also uses an interval of 10° in the *Siddhānta-Śiromaṇi* (apart from the 225' interval). For such large intervals, it is necessary to go beyond linear interpolation. It was Brahmagupta who gave the second-order interpolation formula for finding trignometric functions for arbitrary angles for the first time, in his *Khaṇḍakhādyaka*. Bhāskara also gives this interpolation formula in the *Siddhānta-Śiromaṇi* and explains it too in the *upapatti*. This involves invoking the proportionality principle in a non-trivial manner. We now discuss Bhāskara's statement of the second-sorder interpolation formula and his explanation for the same.

In verse 16 of Spaṣṭādhikāra (chapter on true longitudes), he says in the *Siddhānta-Śiromaṇi*:

> *yātaiṣyayoḥ khaṇḍakayorviśeṣaḥ śeṣāṁśanighno nakhahr̥t tadūnam* ।
> *yutaṁ gataiṣyaikyadalaṁ sphuṭaṁ syāt kramotkramajyākaraṇetra bhogyam* ॥16॥

> The difference of the preceding and succeeding *R*sine differences is multiplied by the remaining degrees and divided by 20 (*nakha*), and this result subtracted from the arithmetic mean of the preceding and succeeding [*R*sine differences] gives the rectified *R*sine difference. In the case of *utkrama-jyā* or *R*versine, the result is added (instead of subtracted).

Bhāskara explains it thus:

> *Upapatti* (Rationale): The *R*sine difference at the midpoint of the preceding and succeeding *R*sine differences should be their arithmetic mean. The succeeding *R*sine difference should be at the end of the interval. [Now, use] the rule of three. If for an interval of 10°, we have half the difference between the two, what should be the difference, for the remainder expressed in degrees? Also, by the rule of three, multiply the remainder of degrees by the difference between the preceding and succeeding *R*sines and divide by 20. Subtract the result from the arithmetic

mean of preceding and succeeding *R*sines, since the differences decrease in the case of *R*sines, and add the result for *R*versines, since the differences increase in the case of *R*versines.

Bhāskara's *upapatti* can be understood thus:

Here Bhāskara gives a second-order interpolation formula for an intermediate angle, which is the same as in the *Khaṇḍakhādyaka* of Brahmagupta conceptually (Sengupta 1934).

Let $i \cdot 10 < \theta < (i+1) \cdot 10°$, i.e. the point on the quadrant is between $i \cdot 10°$ and $(i+1)10°$. Then Bhāskara defines a "rectified" *R*sine difference corresponding to the relevant 10° interval. Let

$$\Delta_i = R\sin(i - 10) - R\sin[(i-1)10],$$

be the tabulated *R*sine difference corresponding to the 10° interval between $(i-1) \cdot 10°$ and $(i - 10°)$. It is obvious that

$$\Delta_{i+1} = R\sin[(i+1)10] - R\sin(i \cdot 10).$$

Then, the rectified *R*sine difference Δ'_{i+1} corresponding to the remainder Ψ within the 10° interval between $i \cdot 10°$ and $(i+1)10°$ is defined as

$$\Delta'_{i+1} = \frac{\Delta_i + \Delta_{i+1}}{2} - \psi \frac{\left(\Delta_i + \Delta_{i+1}\right)}{20}.$$

Let $\theta = i \cdot 10 + \Psi$. Then *R*sin θ obtained using the rectified *R*sine difference is given by

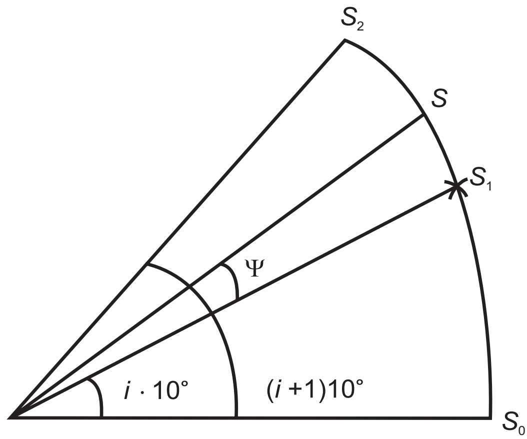


fig. 22.3: Pertaining to the second-order interpolation formula

$$R\sin\theta = R\sin(i\cdot 10) + \psi\frac{\Delta'_{i+1}}{10}.$$

Bhāskara's reasoning for the expression for the rectified *R*sine difference is as follows: The *R*sine difference for the "previous" 10° interval is Δ_i, whereas it is Δ_{i+1} for the "coming" 10° interval S_1S_2. The *R*sine difference at the junction of these two intervals at S_1 is taken to be $\frac{\Delta_i + \Delta_{i+1}}{2}$. The *R*sine difference at the end of the interval S_1S_2 is taken to be Δ_{i+1} itself. As Δ_{i+1} can be written as:

$$\Delta_{i+1} = \frac{\Delta_i + \Delta_{i+1}}{2} - \frac{\Delta_i - \Delta_{i+1}}{2}.$$

The change in the *R*sine difference over the full 10° interval S_1S_2 is given by

$$\frac{\Delta_i + \Delta_{i+1}}{2} - \Delta_{i+1} = \frac{\Delta_i - \Delta_{i+1}}{2}.$$

Then the change in the *R*sine difference at the desired point *S* can be found from the rule of proportions:

$$\frac{\Delta_i + \Delta_{i+1}}{2} - \Delta'_{i+1} = \frac{\psi}{10}\cdot\frac{\Delta_i - \Delta_{i+1}}{2},$$

or

$$\Delta'_{i+1} = \frac{\Delta_i + \Delta_{i+1}}{2} - \frac{\psi\left(\Delta_i - \Delta_{i+1}\right)}{20}.$$

which is the stated result.

Here, the *R*sine difference at the beginning of the interval S_1S_2 is taken to be $\frac{\Delta_i + \Delta_{i+1}}{2}$, which is the mean of two tabulated *R*sine differences, whereas it is taken to be Δ_{i+1}, a tabulated *R*sine difference over the interval $i\cdot 10 < \theta < (i+1)\cdot 10°$. This is a plausible argument. In any case, we have an imaginative use of the rule of proportions here.

Comparison with the Taylor Series up to the Second Order

Writing θ as $\theta = \theta_0 + (\theta - \theta_0)$, where $\theta_0 = i\cdot 10$ and $\Psi = \theta - \theta_0$ and using the expression for the rectified *R*sine difference, Δ'_{i+1} and rewriting in a slightly different form, we have:

$$R\sin\theta = R\sin\theta_0 + (\theta-\theta_0)\left(\frac{\Delta_i+\Delta_{i+1}}{2\cdot 10}\right) + \frac{(\theta-\theta_0)^2}{2}\frac{\left[\dfrac{\Delta_{i+1}}{10}-\dfrac{\Delta_i}{10}\right]}{10}.$$

The Taylor series for R sin θ up to the second order is

$$R\sin\theta = R\sin\theta_0 + (\theta-\theta_0)\frac{d\left(R\sin\theta\right)}{d\theta}\bigg|_{\theta-\theta_0} + \frac{(\theta-\theta_0)^2}{2}\frac{d^2\left(R\sin\theta\right)}{d\theta^2}\bigg|_{\theta=\theta_0}.$$

So Brahmagupta/Bhāskara's expression has the same form with the arithmetic mean of the R sine differences per unit degree in the "previous" and the "coming" intervals,

$$\frac{\Delta_i+\Delta_{i+1}}{2\cdot 10} = \frac{1}{2}\left[\frac{R\sin(i\cdot 10)-R\sin\left[(i-1)10\right]}{10}+\frac{R\sin\left[(i+1)10\right]-R\sin(i\cdot 10)}{10}\right],$$

playing the role of derivative $\frac{d(R\sin\theta)}{d\theta}$ and the rate of change of the Rsine difference per unit interval 1° that is $\frac{1}{10}\left[\frac{\Delta_{i+1}}{10}-\frac{\Delta_i}{10}\right]$ playing the role of the second derivative $\frac{d^2(R\sin\theta)}{d\theta^2}$.

A Relation among the Moon's "Vākyas" (Mnemonics) Using the Proportionality Principle

MOON'S LONGITUDE IN THE VĀKYA SYSTEM

In the *vākya* system of astronomy prevalent in south India, the true longitudes of the sun, the moon and the planets can be found at regular intervals, using *vākya*s (mnemonics) (Sriram 2015; Pai et al. 2016a). These are based on the various periodicities associated with these celestial bodies. For example, moon's anomaly completes very nearly 9 revolutions in 248 days, and correspondingly, there are 248 *candra-vākya*s for the moon, which give the longitudes of the moon at mean sunrise on 248 successive days, beginning with the day at the mean sunrise of which the moon's anomaly is zero. There are more elaborate tables of *vākya*s for the longitudes of planets which involve their zodiacal anomaly, as well as the solar anomaly. We are concerned only with the moon's longitude in the *vākya* system here.

The moon's true longitude is obtained by applying the "equation of centre" to the mean longitude. The equation of centre at any instant depends upon the moon's "anomaly" which is the angular separation between the "mean moon" and the "apogee" of the moon. The *khaṇḍa-dina* is the day at the sunrise of which the moon's anomaly is zero. The *candra-vākya*s are based on the following formula for the change in the true longitude of the moon, *i* days after the *khaṇḍa-dina* (Sriram 2015; Pai et al. 2016a, b):

$$V_i = R_1 \cdot 360 \cdot i - \sin^{-1}\left[\frac{7}{80} \times \sin(R_2 \times 360 \cdot i)\right], \qquad (1)$$

where R_1 and R_2 are the rates of motion of the moon and its anomaly respectively, in revolutions per day. The second term represents the equation of centre of the moon. As it stands, V_i is in degrees. The *candra-vākya*s are essentially the values of V_i, after converting them to *rāśi*s (zodiacal signs), degrees, minutes and seconds, and expressed in the *kaṭapayādi* system.

R_1, the mean rate of motion of the moon, is taken to be $\frac{4909031760}{134122987500} = \frac{1}{27.32167852}$ revolution per day in the texts related to the *vākya* system. It will be seen that the value of R_1 does not play any role in the relation among the *vākya*s that we are considering. For finding the *vararuci-vākya*s (mnemonics due to Vararuci (probably seventh century CE)), R_2 is taken to be $\frac{9}{248}$ revolution per day.[4] For the *mādhava-vākya*s (mnemonics due to Mādhava of Kerala (fourteenth century CE), $R_2 = \frac{6845}{188611}$ revolution per day, used in the *Veṇvāroha* and the *Sphuṭacanrāpti* composed by Mādhava), which is more accurate than $\frac{9}{248}$ (Pai et al. 2016b).

VĀKYAŚODHANA: ERROR CORRECTION CHECKS FOR CANDRA-VĀKYAS[5]

Vararuci-Vākyas

Substituting the value of $R_2 = \frac{9}{248}$ in this case,

$$V_1 = R_1 \cdot 360 \cdot i - \sin^{-1}\left[\frac{7}{80} \times \sin\left(\frac{9}{248} \times 360 \cdot i\right)\right].$$

---

[4] This corresponds to 9 revolutions of the anomaly in 248 days. This anomaly cycle had been noticed by Babylonian, Greek and Indian astronomers of yore.

[5] Pai et al. 2016b; Sriram, 2017.

Hence,

$$V_{248-i} = R_1 \cdot 360 \cdot (248-i) - \sin^{-1}\left[\frac{7}{80} \times \sin\left(\frac{9}{248} \times 360 \cdot (248-i)\right)\right]$$

$$= R_1 \cdot 360 \cdot (248-i) + \sin^{-1}\left[\frac{7}{80} \times \sin\left(\frac{9}{248} \times 360 \cdot i\right)\right],$$

$$V_{248} = R_1 \cdot 360 \cdot 248 - \sin^{-1}\left[\frac{7}{80} \times \sin\left(\frac{9}{248} \times 360 \cdot 248\right)\right]$$

$$= R_1 \cdot 360 \cdot 248,$$

as the last time in the RHS of the equation for V_{248} is 0. Clearly,

$$V_i + V_{248-i} = V_{248} \text{ (modulo } 360°\text{)}. \tag{2}$$

This implies that if there is any doubt about the value of V_i, this relation can be used to find it, if V_{248-i} is known. Hence, it is termed the *vākyaśodhana* (error correction check for mnemonics (for moon)) (Pai et al. 2016b).

Mādhava-Vākyas

In this case, as $R_2 \neq \frac{9}{248}$, the relation (2) clearly does not hold. For the *mādhava-vākya*s, the *vākyaśodhana* procedure is as follows (Pai et al. 2016b):

Suppose one is in doubt about V_i. Let $j = 248 - i$. Then, V_j is the complementary *vākya*. If V_j and the *vākya*s above and below it are known, find:

$$V_j - \frac{(V_{j+1} - V_{j-1} - 2V_1)}{225}.$$

Then,

$$V_i = V_{248} - \left[V_j - \frac{(V_{j+1} - V_{j-1} - 2V_1)}{225}\right],\ j = 248 - i. \tag{3}$$

Note that $i = 248 - j$. We rewrite the above equation in the form:

$$\delta_j \equiv V_j + V_{248-j} - V_{248} = \frac{V_{j+1} - V_{j-1} - 2V_1}{225}. \tag{4}$$

We now show that the above relation is valid to a very good approximation, using the ubiquitous Indian principle of *trairāśika* or the "rule of three" (Sriram 2017).

Explanation of the Vākyaśodhana Expression for δ_j *Using Trairāśika* (Sriram 2017)

We denote the Mādhava value $\frac{6845}{188611}$ for R_2 by α. Then,

$$\begin{aligned}\delta_j &= V_j + V_{248-j} - V_{248} \\ &= R_1 \cdot 360 \cdot j - \sin^{-1}\left[\frac{7}{80}\sin(\alpha \cdot 360 \cdot j)\right] \\ &+ R_1 \cdot 360 \cdot (248 - j) - \sin^{-1}\left[\frac{7}{80}\sin(\alpha \cdot 360 \cdot (248 - j))\right] \\ &- R_1 \cdot 360 \cdot 248 + \sin^{-1}\left[\frac{7}{80}\sin(\alpha \cdot 360 \cdot 248\right].\end{aligned}$$

Therefore,

$$\begin{aligned}-\delta_j = \sin^{-1}&\left[\frac{7}{80}\sin(\alpha \cdot 360 \cdot j)\right] \\ +\sin^{-1}&\left[\frac{7}{80}\sin(\alpha \cdot 360 \cdot (248 - j))\right] \\ -\sin^{-1}&\left[\frac{7}{80}\sin(\alpha \cdot 360 \cdot 248\right].\end{aligned}$$

We split α as

$$\alpha = \left(\alpha - \frac{9}{248}\right) + \frac{9}{248}.$$

Hence,

$$\begin{aligned}-\delta_j &= \sin^{-1}\left[\frac{7}{80}\sin\left(\left(\alpha - \frac{9}{248}\right) \cdot 360 \cdot j + \frac{9}{248} \cdot 360 \cdot j\right)\right] \\ &+ \sin^{-1}\left[\frac{7}{80}\sin\left(\left(\alpha - \frac{9}{248}\right) \cdot 360 \cdot (248 - j) + \frac{9}{248} \cdot 360 \cdot (248 - j)\right)\right] \\ &- \sin^{-1}\left[\frac{7}{80}\sin\left(\left(\alpha - \frac{9}{248}\right) \cdot 360 \cdot 248 + \frac{9}{248} \cdot 360 \cdot 248\right)\right].\end{aligned}$$

Let

$$\in = \left(\alpha - \frac{9}{248}\right) \times 360 = 4.6948 \times 10^{-4}.$$

Using this notation in the above equation we have,

$$-\delta_j = \sin^{-1}\left[\frac{7}{80}\sin\left(\frac{9}{248}\cdot 360\cdot j + \in\cdot j\right)\right]$$
$$-\sin^{-1}\left[\frac{7}{80}\sin\left(\frac{9}{248}\cdot 360\cdot j - \in\cdot(248-j)\right)\right]$$
$$-\sin^{-1}\left[\frac{7}{80}\sin\left(\in\cdot 248\right)\right].$$

Let f be the function representing the equation of centre, $\sin^{-1}$[sin $\frac{7}{80}$ ( )], where ( ) is the anomaly. Hence,

$$-\delta_j = f\left(\frac{9}{248}.360.j + \in.j\right) - f\left(\frac{9}{248}.360.j - \in.(248-j)\right) - f(\in.248)$$
$$= y_1 - f(\in.248),$$
$$= \left(\frac{9}{248}.360.j + \in.j\right) - \left(\frac{9}{248}.360.j - \in.(248-j)\right)$$
$$= \in.248 \equiv x_1, \text{around a value of anomaly} \tag{5}$$

where y_1 is the difference in the equation of centre corresponding to a change in the anomaly (which is the argument) equal to $\left(\frac{9}{248}\cdot 360\cdot j + \in\cdot j\right) - \left(\frac{9}{248}\cdot 360\cdot j - \in\cdot(248-j)\right) = \in \cdot 248 \equiv x_1$, around a value of anomaly equal to $\frac{9}{248} \times 360.\, j$. Note that the change in the anomaly which is proportional to $\in$ is resulting from the departure of $\alpha = R_2$ from $\frac{9}{248}$.

Now consider a different kind of difference:

$$V_{j+1} - V_{j-1} = R_1 \cdot 360 \cdot (j+1) - f(\alpha\, 360 \cdot (j+1))$$
$$- [R_1 \cdot 360 \cdot (j-1) - f(\alpha\, 360 \cdot (j-1))]$$

Hence, $$V_{j+1} - V_{j-1} = 2 \cdot R_1 \cdot 360 - y_2, \tag{6}$$

where y_2 is the difference in the equation of centre corresponding to a change in the anomaly equal to $(\alpha \cdot 360 \cdot (j+1)) - (\alpha \cdot 360 \cdot (j-1)) = 2 \cdot \alpha \cdot 360 \equiv x_2$, around a value of anomaly equal to $\alpha \times 360 \cdot j$. Here, the change in the anomaly is due to the fact that we are considering the *vākya*s for two different days, corresponding to $j+1$ and $j-1$.

y_1 and y_2 are the changes in the equation of centre corresponding to changes in the anomaly equal to x_1 and x_2 respectively. Now, we use the *trairāśika* (the rule of three), or the law of proportions,[6] which plays such an important role in Indian mathematics and astronomy:

$$y_1 : x_1 = y_2 : x_2,$$

or

$$y_1 = \frac{y_2}{x_2} \cdot x_1. \tag{7}$$

Using equations (5), (6) and (7), and the values of $x_1 = \in \cdot 248$ and $x_2 = 2 \cdot \alpha \cdot 360$, we have,

$$\delta_j = -y_1 + f(\in \cdot 248)$$

$$= \frac{(V_{j+1} - V_{j-1}) - 2 \cdot R_1 \cdot 360}{2 \cdot \alpha \cdot 360} \times \in \cdot 248 + f(\in \cdot 248).$$

Now,

$$\frac{\in \cdot 248}{2 \cdot \alpha \cdot 360} = \frac{4 \times 6948 \times 248 \times 188611}{2 \times 360 \times 6845} \times 10^{-4}$$

$$= 4 \times 4558 \times 10^{-3} = \frac{1}{224.4244}.$$

This is approximated as $\frac{1}{225}$. Therefore,

$$\delta_j \approx \frac{V_{j+1} - V_{j-1} - 2X}{225}, \tag{8}$$

where,

$$X = R_1 \cdot 360 - \frac{f(\in \cdot 248)}{2} \times 225. \tag{9}$$

Now,

$$\frac{f(\in \cdot 248)}{2} \times 225 = \frac{1}{2} \sin^{-1}\left(\frac{7}{80} \sin(248 \cdot \in)\right) \times 225 = 1.1449.$$

Hence,

$$X = R_1 \cdot 360 - 1.1449. \tag{10}$$

---

[6] Actually, x_1 is the change in the anomaly around $\frac{9}{248} \cdot 360 \cdot j$, whereas x_2 is the change in the anomaly around $\alpha \cdot 360 \cdot j$. As $\alpha = \frac{6845}{188611} \approx \frac{9}{248}$, we ignore this difference, which will lead to changes of higher order in $\in$.

From equation (1),

$$V_1 = R_1 \cdot 360 - \sin^{-1}\left(\frac{7}{80}\sin\left(\frac{6845}{188611}\times 360\right)\right)$$

$$= R_1 \cdot 360 - 1.1334\,. \tag{11}$$

Comparing equations (10) and (11), we find:

$$X \approx V_1. \tag{12}$$

Substituting this in equation (8), we have:

$$\delta_j = V_j + V_{248-j} - V_{248} \approx \frac{V_{j+1} - V_{j-1} - 2V_1}{225}, \tag{13}$$

which is the desired result.

It is very significant that such a highly non-trivial relation among the *candra-vākya*s (moon's menmonics) can be derived by a judicious application of *trairāśika* (rule of three).

References

Arkasomayaji, 2000, *Siddhānta-Śiromaṇi of Bhāskarācārya-II, Grahagaṇitādhyāya*, translation of the text with explanations, Triupati: Rashtriya Sanskrit Vidyapeetha.

Bhāskarācārya's Līlāvatī with Colebrooke's translation and notes by H.C. Banerjee, 2nd edn, Calcutta: The Book Company, 1927: reprint, New Delhi: Asia Educational Series, 1993.

Bhāskarācārya's Līlāvatī, tr. and ed. A.B. Padmanabha Rao, Ernakulam: Chinmaya International Foundation Shodha Sansthan, Part I, 2015, Part II, 2014.

Pai, R. Venketeswara, M.S. Sriram, K. Ramasubramanian and M.D. Srinivas, 2016a, "An Overview of the Vākya Method of Computing the Longitudes of the Sun and Moon", in *History of Indian Astronomy : A Handbook*, ed. K. Ramasubramanian, Aniket Sule and Mayank Vahia, pp. 430-61, Mumbai: IIT Bombay and TIFR, Mumbai.

Pai, R. Venketeswara, K. Ramasubramanian, M.D. Srinivas and M.S. Sriram, 2016b, "The Candravākyas of Mādhava", *Gaṇita Bharati*, 38(2): 111-39.

Sengupta, P.C., 1934, *The Khaṇḍakhādyaka: An Astronomical Treatise of Brahmaguupta*, tr. into English with an introduction, notes, illustrations and exercises, Calcutta: University of Calcutta.

Siddhānta-Śiromaṇi of Bhāskarācārya with his *Vāsanābhāṣya*, ed. Bāpū Deva Śāstrī, revised by Gaṇapati Śāstri, 4th edn, Varanasi: Chaukhambha Sanskrit Sansthan, 2005.

Sriram, M.S., 2015, "Vākya System of Astronomy", in *Handbook of Archeoastronomy and Ethnoastronomy*, ed. C.L.N. Ruggles, pp. 1991-2000, New York: Springer Science.

Sriram, M.S. 2017, "Explanation of the Vākyaśodhana Procedure for the Candravākyas", *Gaṇita Bharati*, 39(1): 47-53.

23

Śuddhadṛggaṇita
An Astronomical Treatise from Northern Kerala

Anil Narayanan

Abstract: The present paper analyses a modern Keralite astronomical work – *Śuddhadṛggaṇita*. This treatise written in Sanskrit, authored by V.P.K. Potuval, has been published from the Jyotisadanam of Payyanur, Kerala. The present paper discusses the methods therein to find the Kali epoch and the mean position of a planet. It summarizes how *Śuddhadṛggaṇita* serves in maintaining the continuity of the tradition of the Kerala school of astronomy and mathematics.

INDIAN mathematics encompasses the era of the Kerala School of Mathematics. The Kerala School of Mathematics flourished between fourteenth and eighteenth century. During this age, Kerala immensely contributed to the field of mathematics. It is justifiably claimed as the golden period in the history of Indian mathematics (Parameswaran 1998: iii). To our knowledge, the *guru–śiṣya paramparā* of the Kerala School of Mathematics commences from Saṅgamagrāma Mādhava[1] (1340–1425 CE). His decisive steps

---

[1] For more details on the contribution of Mādhava, refer Sarma 1972: 15-17; Bag 1976: 54-57; Gold and Pingree 1991; Gupta 1973, 1975, 1976, 1987, 1992 and Hayashi, Kusuba, Yano 1990, etc.

were followed by Vaṭaśśeri Parameśvaran Nampūtiri[2] (1360–1455 CE), Dāmodara[3] (1410–1520 CE), Keḷallur Nilakaṇṭha Somayājin[4] (1444–1545 CE), Jyeṣṭhadeva[5] (1500–1610 CE), etc. In attempting to solve astronomical problems, the Kerala School of Mathematics independently created a number of important mathematics concepts. Many of the findings of the Kerala School of Mathematics anticipated the discoveries of mathematicians like James Gregory, Newton and Leibnitz. However, the discoveries of the Kerala School of Mathematics were "re-discovered" very late and thanks to the painstaking efforts of T.A Saraswati Amma,[6] K.V. Sarma,[7] C.T. Rajagopal, K. Mukundamarar (Rajagopal and Rangachari 1978, 1986), etc. for their valuable contributions. K.V. Sarma has rightly evaluated that the spirit of enquiry, stress on observation and experimentation, concern for accuracy, researcher's outlook, and continuity of tradition are some of the salient features of the Kerala School of Mathematics (1972: 7-10). Apart from the aforesaid characteristics, one of the hallmarks of the Kerala tradition is the periodical revision of systems of computations. Many astronomers and mathematicians of the Kerala School of Mathematics introduced refinements and improvements on the methods of calculations, and it indeed paved the way for the development of

---

2 For more details on the contribution of Parsmeśvara, refer Gupta 1977, 1979; Plofkar 1996; Raja 1963.

3 Dāmodara was the son of Parameśvara. No full-fledged work of Dāmodara has come to light. Somayājin has quoted Dāmodara on many occasions in his *Āryabhaṭīya* commentary.

4 For more details, on the contribution of Nīlakaṇṭha, refer Hayashi 1999; Roy 1990; Sarma, Narasimhan and Somayāji 1998.

5 Suggested readings for Jyeṣṭhadeva are Sarma 2008; Divakaran 2011; Sarma and Hariharan 1991.

6 For more details, ref. https://www.insa.nic.in/writereaddata/UpLoadedFiles/IJHS/Vol38_3_8_Obituary.pdf.

7 K.V. Sarma has authored sixty books and 145 research papers. The complete bibliography of the writings of K.V. Sarma on Indian culture, science and literature has been compiled and published from Sri Sarada Education Society Research Centre, Adayar, Chennai, in 2000.

the twin disciplines – astronomy and mathematics. The present paper addresses the feature of continuity of astronomical/mathematical tradition in north Kerala by examining a modern manual, called *Śuddhadṛggaṇita*[8], into account.

Continuity of Tradition and Periodical Revision in North Kerala in the Medieval Period

Kerala astronomers and mathematicians adhered to the Āryabhaṭīyan system and followed the *Āryabhaṭīya*. The Kerala School of Mathematics has produced a large number of commentaries on the *Āryabhaṭīya*.[9] But at the same time, they were also deeply engaged in revising, supplementing and correcting the Āryabhaṭīyan system for more accurate results. The systems of computations were revised periodically. One of the significant events in the annals of Kerala astronomy is the revision of the Āryabhaṭīyan system of calculation by Haridatta (*c*.683 CE). Through his works, the *Grahacāranibandhana* (a digest on the motion of the planets) (Sarma 1954) and the *Mahāmārganibandana* (a digest of extensive full-fledged astronomy) (Sarma 1954: 5), Haridatta promulgated the *parahita* system of calculation. Tradition (Parameswaryyar 1998) holds that the system was proclaimed on the occasion of the twelve-year Māmāṅkaṁ (Skt. Mahāmāgham) festival, at Tirunavaya in north Kerala in 683 CE. These corrections were called *bhaṭasaṁskāra* (corrections to Āryabhaṭa). It was also called *śakābdda-saṁskāra* since it applied from the date of Āryabhaṭa in the Śaka year 444, at which date his constants gave accurate results. The *Bhaṭasaṁskāra* specifies that for every completed year after Śaka 444, a correction in minutes (*kalā*) – 9/85, – 65/134, –

[8] This twentieth-century astronomical work authored by V.P.K. Potuval from the Payyanur, Kannur, north Kerala. The text has been published with an autocommentary in Malayalam.

[9] Parameśvara's commentary (available at https://ia800208.us.archive.org/1/items/aryabhatiyawithc00arya/aryabhatiyawithc00arya.pdf), Keḷallur Nilakaṇṭhasomayaji's comm.(available at https://ia601902.us.archive.org/28/items/Trivandrum_Sanskrit_Series_TSS/TSS-101_Aryabhatiya_With_the_Commentary_of_Nilakanta_Somasutvan_Part_1_-_KS_Sastri_1930.pdf etc. are notable.

13/32, + 45/235, + 420/235, – 47/235, – 153/235, + 20/235 should be made to the mean positions of the moon, moon's apsis, moon's node, Mars, Mercury, Jupiter, Venus and Saturn respectively (ibid.). Haridatta also advocated that no correction is necessary in the case of the sun (ibid.). Inspired by the works of Haridatta, during later times, treatises like the *Grahacāranibandhanasaṁgraha* (Sarma 1954: App.) were composed. Through the course of years, the results of computation began to differ appreciably from those of actual observation. This necessitated corrections to the *parahita* system and Vaṭaśśeri Parameśvaran Nampūtiri (henceforth Parameśvara) was prompted to compose his magnum opus the *Dṛggaṇita*. The revealing statements of Parameśvara at the very outset of his work are as follows:

> (The positions of) planets derived according to the *parahita* (system of computation) are found to be different (from their actual positions) as seen by the eye. And, in the authoritative texts (*śāstra*) it is said that (only) positions as observed (should be taken) as the true ones. (The positions of) the planets are the means of knowing the times specified for (the performance of) meritorious acts. (Here), times calculated from incorrect (positions of) planets will not be auspicious for those acts. Hence, efforts should be made for knowing the true (positions of) planets by those who are learned in the sciences and by those who are experts in spherics.[10]

The *Dṛggaṇita* of Parameśvara has two parts and the first part consists of four sections called *paricchedas*. The method of calculation of days elapsed in the Kali epoch and the methods for the computation

[10] Translation by K.V. Sarma of the verses:

दृश्यन्ते विहगा दृष्ट्या भिन्ना परहितोदिताः।
प्रत्यक्षसिद्धाः स्पष्टाः स्युर्ग्रहाः शास्त्रेष्वितीरितम्।।

सत्कर्मोदितकालस्य ग्रहा हि ज्ञानसाधनम्।
अस्पष्टविहगैः सिद्धः कालः शुद्धो न कर्मणि।।

ये तु शास्त्रविदस्तद्वद् गोलयुक्तिविदश्च तैः ।
स्फुटखेचरविज्ञाने यत्नः कार्यो द्विजैरतः ।।

of the mean positions of the planets are discussed in the first *pariccheda*. The position of the mean planets at the commencement of the Kali epoch have been discussed in the second *pariccheda*. The computation of the true position of planets is dealt with in the third, and the fourth *pariccheda* is on the derivation of the sine of arc of anomaly and commutation (*manda-jyā* and *śīghra-jyā*) and on the method for the calculation of the arc from the sine. The second part of the text appears to be a reiteration of part one. But the difference is that the reiteration is done by making use of the *kaṭapayādi* system.[11] The author himself has stated that the purpose of reiteration is "for the benefit of young learners".[12] As the results obtained in the *Dṛggaṇita* system was found more accurate, it was used for horoscopy (*jātaka*), astrological queries (*praśna*s) and for the computation of eclipses (*grahaṇa*), whereas the use of *parahita* was confined to only fixing the auspicious time for rituals and ceremonies (*muhūrta*).

Traditional astronomers and astrologers of north Kerala followed the *parahita* system for their calculations up to the first three decades of the twentieth century. Then, some revolutionary changes took place. Optical instruments like telescope became common for observation. These instruments are of great use for observing remote planets by collecting electromagnetic radiation such as visible light. With the help of telescopes and artificial satellites, the positions of planets were located more accurately. Hence, it was felt by the traditional astrologers that the positions of planets as given by modern science (with the help of satellites, etc.) can be taken into account and further calculations can be carried out in the traditional manner itself. This resulted in the advent of a new system of computation called *Śuddhadṛggaṇita*.[13] In Kerala,

---

[11] For more details and applications of the *kaṭapayādi* system, see Narayanan (2013).

[12] स्पष्टीकर्तुं दृग्गणितं वक्ष्ये कटपयादिभिः । उक्तमर्थे परहितं बालाभ्यासहितं च तत् ॥
— दृग्गणितम्, द्वितीयो भागः, प्रथमः श्लोकः

[13] In the Indroduction of *Śuddhadṛggaṇita*, it is stated that the system of *Śuddhadṛggaṇita* was first suggested by a north Indian, Veṅkateśaketakara, through his work *Jyotirgaṇita* (Śaka 1812).

Puliyur Purushottaman Namputiri,[14] K.V.A. Ramapotuval[15] and V.P.K. Potuval[16] took initiatives for implementing the system of *Śuddhadṛgganita*. Among these three, it was V.P.K. Potuval who first introduced the system in northern Kerala by composing the work called *Śuddhadṛgganita.*

Types of Astronomical Manuals and the Nature of Śuddhadṛgganita

E. Sreedharan, in his Introduction to *Śuddhadṛgganita,* has mentioned about the different types of astronomical manuals. All the primary astronomical manuals can be grouped into four classes or types. The first class consists of the Siddhānta texts. These types of texts include very long procedures for computations. For calculating the mean position of planets, computations have to be done from the starting date of the first *kalpa* and need to be carried over to the desired date.[17] Most of the ancient texts like the *Brāhmasphuṭa-Siddhānta* and *Siddhānta-Śiromaṇi* come under this class. The second class of texts is called Tantra texts in which calculations from the current *yuga* up to the desired date are necessary to derive the mean position of planets. Hence, the calculations prescribed by Tantra texts is simpler compared to the Siddhānta texts. Texts like the *Āryabhaṭīya, Tantrasaṁgraha* and

---

[14] Puliyur suggested the system through his work *Gaṇitanirṇaya* and the text was used in the southern Kerala.

[15] Through *Gaṇitaprakāśikā*, K.V.A. Potuval suggested the system and it gained popularity in northern Kerala.

[16] V.P.K. Potuval is the author of the text *Śuddhadṛgganita.* The text was composed in 1978 CE. Potuval hails from the Payyannur area of Kannur – a north Kerala district. Apart from *Śuddhadṛgganita*, he has another work called *Sūkṣmadṛgganitasopāna* to his credit. He presented his *Śuddhadṛgganita* scheme of computation in an august assembly of astronomers and astrologers at Ayodhyā and was awarded the title *Jyotirbhūṣaṇam*.

[17] यस्मिन् कल्पादेरारभ्य गताब्दमासदिनादेः सौरसावनचान्द्रमानान्यवगम्य सौरसावनगताहर्गणानां मध्यमादीनां च कर्म उच्यते, तत् सिद्धान्तलक्षणम् इति केतकीग्रहगणितभाष्यम् ।।

Yuktibhāṣā, are examples of the Tantra type of texts.[18] The third class is known as the Karaṇa texts. Here, for finding the mean position or for finding the Kali epoch, the calculations are carried over from a *karaṇārambhadina* (which will be suggested by the author) to the desired date. Hence, practically the simplest method of calculation is the one suggested in the Karaṇa type of texts. The *Khaṇḍakhādyaka*, *Karaṇakutūhala*, etc. are Karaṇa type of texts.[19] Most of the texts produced by the Kerala School of Mathematics are Karaṇa texts. The *Grahacāranibandhana*, *Dṛggaṇita Pañcabodha*, *Ṣaḍratnamālā*, *Karaṇapaddhati*, *Jyotiṣśāstrasubodhinī*, etc. come under the Karaṇa class. The fourth class is known as Vākya texts, in which mnemonics are organized into tables. So, a person without much knowledge of mathematics can find planetary positions, without doing much calculations. The *Vararūcivākya*, *Vākyakaraṇa*, *Kujādipañcagrahavākya*s, etc. are examples of the Vākya class of texts. As *SDG* suggests a *karaṇārambhadina* for calculations, it is Karaṇa type of text.

Topics such as finding the *ahargaṇa* (Kali epoch), finding the mean position of planets, finding the true position of planets, are generally discussed in the Karaṇa type of texts. As stated, they also provide a *karaṇārambhadina* (a date, starting from which all the calculations are carried over) and the position of planets at a specified date and at a specific time (which are known as *dhruvaka*s).

Parameters Used in Śuddhadṛggaṇita

Śuddhadṛggaṇita, being a Karaṇa type of text, suggests methods for finding the Kali epoch, mean position of planets, true position of planets, etc. by providing a *karaṇārambhadina*. *Karaṇārambhadina*s provided by the Karaṇa texts are in order to make the calculations easy. The *karaṇarambhadina* suggested by *Śuddhadṛggaṇita* is the Independence day of our country, i.e. 15 August 1947, and the desired time given is the sunrise of the same day. The calculations and positions of planets provided in *Śuddhadṛggaṇita* are in accordance with the place Trivandrum, the capital of Kerala

---

[18] वर्तमानयुगादेर्वर्षाण्येव ज्ञात्वा उच्यते, तत् तन्त्रम् ।।

[19] वर्तमानशकमध्ये अभीष्टदिनादारभ्यैव ज्ञात्वा उच्यते तत् करणलक्षणम् ।।

(having a longitude of 77°E). Hence, it should be noted that the position of planets at a longitude 77°E on 15 August 1947 at sunrise are directly provided in *Śuddhadṛggaṇita*. These mean positions of planets on a desired date and at a desired time suggested by Karaṇa texts are called *dhruvaka*s. To get the positions of planets at any other date, time and place, further calculations have to be carried out.

Śuddhadṛggaṇita on Finding the Kali Epoch

Suppose one has to find the *kali-dina-saṅkhya* of any day, say the 1st day of the month of Siṁha in the Kollam[20] year 1175 (i.e. 17 August 1999, Tuesday).

Then according to *Śuddhadṛggaṇita*, one has to proceed as follows:[21]

Step 1: Multiply the Kollam year (to which the Meṣa month of the target date belongs) with 365 – i.e. 1174 × 365 = 4,28,510. This result is known as *diavasa-saṅkhyā.*

Step 2: The Kollam year is multiplied by 10 and divided by 39, and the result is added to the obtained *divasa-saṅkhya* – i.e. (1174 × 10)/39 + 428,510 = 428,811.

Step 3: The number 1,434,007 is added to the final result obtained in step 2, i.e. 428,811 + 1,434,007 = 1,862,818. This will be the *ahargaṇa* of the 1st day of Meṣa (Aries) of the Kollam year we have considered.

Step 4: The number of days elapsed after the 1st day of Meṣa up to the target date is added to the result in step 3, i.e. 125 is added and hence the answer is 1,862,943.

Hence, according to *Śuddhadṛggaṇita*, 1,862,943 is the *kali-dina-saṅkhya* of 1st day of Siṁha month, 1175.

---

20 The Kollam (Kolamba, Skt.) year commenced from 15 August 824 CE. For more details, see Sarma 1996.

21 कोलम्बवर्षाहितमातलिः (365) स्यात् कोलम्बतो धूलि (39) हृतैर्नय (10) घ्नात् । दिनैश्च सेना–नव–गूढयानै (14300007) र्युतोच्छवारात् किलवासरौघः ।।

– *Śuddhadṛggaṇita*, ग्रहमध्यमप्रकरणम् , कारिका १

Śuddhadṛggaṇita on Finding the Mean Position of Planets

Let us now analyse the method of finding the mean position of Sun as suggested by *Śuddhadṛggaṇita*. The following steps are involved in the calculation.

Finding the mean position of the sun as elucidated by *Śuddhadṛggaṇita*:

Step 1: At first, find the difference between the two *ahargaṇa*s, i.e. the *ahargaṇa* (*kali-dina-saṅkhyā*) of *karaṇārambhadina* and the *ahargaṇa* of the desired date. The result obtained is known as *khaṇḍaśeṣa*

Step 2: This *khaṇḍaśeṣa* is multiplied by 11 and divided by 764 to get the *bhāgādi* (*bhāga* means degree so the result should be in degree, minute and second).

Step 3: The *bhāgādi* (obtained in step 2) is subtracted from *khaṇḍaśeṣa*. The result is known as the *prathama phala* of *sūryagati*.

Step 4: *Khaṇḍaśeṣa* is divided by 2,374 and the quotient is known as *dvitīya phala* which will be in *kalādi* (minute, second and arc seconds).

Step 5: The *prathamaphala* obtained in step 3 and *dvitīyaphala* obtained in step 4 is summed up and *sūryagati phala* is found out from this sum.

Step 6: The *sūryagati phala* obtained in step 5 is added with the *sūryasphuṭa* of *karaṇārambhadina* which will give the mean position of the sun (at Trivandrum) at sunrise on the target date.

Now, suppose one has to find out the mean position of the sun on a desired/target date, say the 1st day of the month Siṁha, in the Kollam era 1175. Then, according to *Śuddhadṛggaṇita*:

Step 1: The difference between the *ahargaṇa*s of *karaṇārambhadina* and the desired date has to be found out.

The *ahargaṇa* of the desired date (1st Siṁha of 1175) = 1,862,942. The *ahargaṇa* of *karaṇārambhadina* (15 August 1947) = 1,843,947.

Their difference is 1,862,942 – 1,843,947 = 18,995; which is known as *khaṇḍaśeṣa*.

Step 2: The *khaṇḍaśeṣa* is multiplied by 11 and divided by 764 to get the *bhāgādi*.

18,995 × 11/764 = 273 degrees 29 minutes 18 seconds.

Step 3: The *bhāgādi* (obtained in step 2) is subtracted from *khaṇḍaśeṣa.*

18,995 deg. 00 min. 00 sec. —
273 deg. 29 min. 18 sec.

18,721 deg. 30 min. 42 sec., which is known as the *prathama phala* of *sūryagati*.

Step 4: The *khaṇḍaśeṣa* is divided by 2,374 and the quotient is known as *dvitīya phala* (which will be in *kalādi*/minutes)

$\frac{18,995}{2,374} = 0$ minutes 08 seconds 00 arc seconds.

Step 5: The *prathama phala* and *dvitīya phala* are summed up and the *sūryagati phala* is found out from this sum.

18,721 deg. 30 min. 42 sec. +
0 deg. 08 min 0 sec.

18,721 deg. 38 min. 42 sec. 1 deg. 38 min. 42 sec. (as 18,720 is exactly divisible by 60).

Step 6: The *sūryagati phala* obtained in step 5 is added with the *sūryasphuṭa* of *karaṇārambadina,* which gives the mean position of the sun on the desired date.

The *sūryasphuṭa* of *karaṇārambadina* is provided by *Śuddhadṛggaṇita* by the phrase *mābandhuśrīdharolaṁ*[22] (which in *kaṭapayādi* corresponds to 3 *rāśi* 29 deg. 29 min. 35 sec.)

0 *rāśi* 01 deg. 38 min. 42 sec. +
3 *rāśi* 29 deg. 29 min. 35 sec.

4 *rāśi* 01 deg. 08 min. 17 sec.; which is the mean position of the sun on 1st Siṁha 1175 at sunrise at Trivandrum.

[22] The *grahasphuṭa*s in *karaṇārambhadina* are given in *Śuddhadṛggaṇita* by the verse:

माबन्धुश्रीरधरोलं मणिचयदनुगः सन्निधाविन्दिरेन्दोः पुण्याभिज्ञो मनुष्यस्तपनहिमकरोच्चोरगाणां
ध्रुवा स्युः । भूपालम्बोमरज्ञः खनिगणपुरगः पुण्यतत्त्वोनयार्थी प्राज्ञाचारात्मयोगी कुनवनतकुलो
भौमतश्चात्र सूक्ष्मा: ।।

Conclusion

The methods of finding the *kali-dina-saṅkhyā* and the mean position of planets were among the major subjects of discussion in the Keralite astronomical texts.[23] For example, the seventh-century text *Grahacāranibandhana* of Haridatta had discussed the method of finding *kali-dina-saṅkhyā* and the mean position of planets. Later, in the fourteenth century, the *Dṛggaṇita* of Vaṭaśśeri Parameśvaran Nampūtiri, also discussed the method of finding *kali-dina-saṅkhyā*[24] and the methods of finding the mean positions. But each time when these methods were promulgated, there was some novelty and this novelty does not lie in the methodology. Rather, the novelty lies in the revision of astronomical constants. As the position of planets derived according to some specified system of computation was found to be different from their actual positions, different texts and systems of computations were produced in Kerala periodically. Thus, the contributions of texts like *Śuddhadṛggaṇita* do not lie in the enunciation of any new working methodology but on the periodical revision of different astronomical constants. As has been discussed, *Śuddhadṛggaṇita* being a Karaṇa type of text, made the computations easier by suggesting new astronomical constants. Thus, by suggesting new multipliers and divisors for the derivation of days in the Kali epoch for the calculation of mean position of planets and by revising the systems periodically, *Śuddhadṛggaṇita* serves to maintain the continuity of the Kerala tradition of astronomy and mathematics.

---

[23] Even non-Keralite works have also discussed *kali-dina-saṅkhyā-nayana*. e.g. Śrīpati (eleventh century CE), in his work *Siddhāntaśekhara*, has discussed seven different methods for finding the *kali-dina-saṅkhyā.* For more details refer the *Śekharavaiśiṣṭyam,* Ramakrisha Pejjathaya, SMSP Sanskrit Research Centre, Udupi, pp. 33-36, 2002.

[24] शाकाब्दान् नवनगकुत्रिभिर्युतान् (३१७९) भूषडब्धिविधुनिहतान् (१४६१)
निजनगसप्तनखाम्बुधि(४२०७७)भागयुतानब्धि(४)भिर्हरेलब्धम् ॥

द्युगुणो मध्ये विषुवति भृगुसुतवारोदयादिः स्यात् ।
दिवसद्वयेन हीनः पुनस्तु सः स्फुटिवषुवति स्यात्॥

चात्रादितिथिसमेतो विषुवित्तिथिविरहितः स एव पुनः ।
तिथिषष्ट्यंशविहीनो द्युगुणोभीष्टे दिने भवति ॥ – *Dṛggaṇita*, vv 7-9

References

Bag, A.K., 1976, "Mādhava's Sine and Cosine Series", *Indian J. History Sci.*, 11(1): 54-57.

Divakaran, P.P., 2012, "Birth of Calculus with Special Reference to Yuktibhasha", *Indian Journal of History of Science*, vol 47(4) p: 771-832.

Gold, D. and D. Pingree, 1991, "A Hitherto Unknown Sanskrit Work Concerning Mādhava's Derivation of the Power Series for Sine and Cosine", *Historia Sci.*, 42: 49-65.

Gupta, R.C., 1973, "The Madhava–Gregory Series", *Math. Education*, 7: B67-B70.

———, 1975, "Mādhava's and Other Medieval Indian Values of Pi", *Math. Education,* 9(3): B45-B48.

———, 1976, "Madhava's Power Series Computation of the Sine", *Ganita,* 27(1-2): 19-24.

———, 1977, "Parameśvara's Rule for the Circumradius of a Cyclic Quadrilateral", *Historia Math,* 4: 67-74.

———, 1979, "A Mean-Value-Type Formula for Inverse Interpolation of the Sine", *Ganita,* 30(1-2): 78-82.

———, 1987, "Madhava's Rule for Finding Angle between the Ecliptic and the Horizon and Aryabhata's Knowledge of It", in *History of Oriental Astronomy,* pp. 197-202, Cambridge: Cambridge University Press.

———, 1992, "On the Remainder Term in the Mādhava-Leibniz's Series", *Ganita Bharati*, 14(1-4): 68-71.

Hayashi, T., 1999, "A Set of Rules for the Root-extraction Prescribed by the Sixteenth-century Indian Mathematicians, Nilakantha Somastuvan and Sankara Variyar", *Historia Sci.,* 9(2): 135-53.

Hayashi, T., T. Kusuba and M. Yano, 1990, "The Correction of the Mādhava Series for the Circumference of a Circle", *Centaurus,* 33(2-3): 149-74.

Narayanan N., Anil, 2013, *Some Systems of Computations among the Traditional Astrologers of Kerala*, Calicut: Darsanasudha, University of Calicut Publication Series.

Parameswaran, S., 1998, *The Golden Age of Indian Mathematics*, Kochi: Swadeshi Science Movement.

Parameswaryyar, S. Ullur, 1998, *Keralasahityacaritram*, vol. III, Trivandrum: University of Kerala.

Plofker, K., 1996, "An Example of the Secant Method of Iterative Approximation in a Fifteenth-century Sanskrit Text, *Historia Math.*, 23(3): 246-56.

Raja, K.K., 1963, "Astronomy and Mathematics in Kerala", *Brahmavidya*, 27: 136-43.

Rajagopal, C.T. and M.S. Rangachari, 1978, "On an Untapped Source of Medieval Keralese Mathematics", *Arch. History Exact Sci.,* 18 : 89-102.

———, 1986, "On Medieval Keralese Mathematics", *Arch. History Exact Sci.,* 35: 91-99.

Roy, R., 1990, "The Discovery of the Series Formula for π by Leibniz, Gregory and Nilakantha," *Math. Mag.*, 63(5): 291-306.

Sarma, K.V. (ed.), 1954, *Grahacāranibandhana: A Parahita-Gaṇita Manual*, Madras: Kuppuswami Sastri Research Institute.

———, 1972, *A History of the Kerala School of Hindu Astronomy*, Hoshiarpur: Visweshvarananda Institute.

———, 1996, "Kollam Era", *IJHS*, 31(1).

———, 2008, *Gaṇitayuktibhāṣā of Jyeṣṭhadeva*, New Delhi: Hindustan Book Agency.

Sarma, K.V. and S. Hariharan, 1991, "Yuktibhasa of Jyesthadeva: A Book of Rationales in Indian Mathematics and Astronomy – An Analytical Appraisal", *Indian J. Hist. Sci.,* 26(2): 185-207.

Sarma K.V., V.S. Narasimhan and N. Somayaji, 1998,"Tantrasaṁgraha of Nilakantha Somayaji (Sanskrit, English Translation)", *Indian J. Hist. Sci.,* 33(1), Supplement.

24

कालनिरूपणम्

मुरलिः एस्

त्रिस्कन्धरूपेण विभागिते अस्मिन् ज्योतिषे महाशास्त्रे अयं गणितविषयः सप्रमाणं सुस्पष्टं च उपदिष्टः वर्तते। संस्कृतवाङ्मये गणितस्य मूलं ज्योतिषशास्त्रं भवति। ज्योतिषे विद्यमानस्य सिद्धान्तस्कन्धस्यैव अपरं नाम गणितमिति। तदुक्तं भास्कराचार्येण सिद्धान्तशिरोमणौ –

सिद्धान्तः स उदाहृतोऽत्र गणितस्कन्धप्रबन्धो बुधैः। इति।

एवञ्च सिद्धान्तग्रन्थे कालाधारिताः विचाराः प्रधानतया विचरिताः वर्तन्ते। तदुक्तं भास्कराचार्येण ।

त्रुट्यादि प्रलयन्तकालकलना ... इति।

अत्रोक्त कालस्य एवं गणितस्य च कः संबन्धः काले गणितं कथं भवति इति अस्मिन् कालनिरूपण विषये विचारयामः। कलयति इति कालः। अर्थात् गणयति इत्यर्थः। यथा व्याकरणे अक्षराणां प्रक्रियाप्रयोगादि दशाभेदाः सन्ति। तथा अत्रापि प्रक्रिया प्रक्रियाफलमिति वर्तते। गणितेन निष्पन्नं फलं एव कालः। प्रक्रियां विना कालस्य प्रतिपादनं कर्तुं न शक्यते। तदृशः अयं कालः द्वेधा विभज्यते। महाकालः खण्डकालः इति। महाकालः नित्यः विभुः अनन्तः च भवति। व्यावहारिकः खण्डकालः संवत्सरमासादयः। लोके यानि वस्तूनि कादिलान्तरूपेण व्यवह्रियन्ते तानि अतिशयोक्तानि असाधारणानि च भवन्ति। उदाहरणार्थं तु काली देवता सङ्कल्पः। कादिलान्तपदेन व्यवह्रियमाणा एषा महामाया सर्वाधारमयी एवं सर्वप्राणधात्री च भवति। सर्वेऽपि देवताः तां स्तुत्वा एव लोकरक्षणे प्रवर्तन्ते। किं बहुना मनुष्याः अपि यदा स्वजीवनं पराधीनं वर्तते अथवा दुर्भाग्यशालिनः वा भवन्ति तदा कादिलान्तं कालमेव कारणं इति चिन्तयन्ति। एवं विभिन्न महत्वयुक्त कालोपदेशावसरे सर्यसिद्धान्ते सूर्यांशपुरुषः वदति।

दद्यां कालाश्रयं ज्ञानं ग्रहाणां चरितं महत् इति।

एवमपि

लोकानामन्तकृत् कालः कालोऽन्यः कलनात्मकः।
सद्विधा स्थूलसूक्ष्मत्वात् मूर्तश्चामूर्त उच्यते।

अर्थात् सर्वलोकानां निर्णयकृत् कालः महाकालः अमूर्तः एक एवः। कलनात्मकः अथवा गणनात्मककालः मूर्तः कालः द्वितीयः। एतादृशः कलनात्मकः कालः द्वेधा भवति। तदुच्यते सूर्यसिद्धान्ते

प्रणादिः कथितो मुर्तः त्रुट्याद्यः अमूर्तसंज्ञकः ।
षड्भिः प्राणैः विनाडी स्यात् तत् षष्ट्या नाडिका स्मृता ।।

प्राणादयः गणनार्हाः व्यवहर्तुं योग्याः मूर्ताः स्थूलाः च भवन्ति। त्रुट्यादयः कालाः गणनार्हेऽपि व्यवहर्तुं अनर्हाः अमूर्ताः च भवन्ति। तादृशकालास्तु।

सूच्या पद्मपत्रभेदनकालः त्रुटिः इति अभिधीयते।

६० त्रुट्यः १ रेणुः	१/५००४ सेकण्ड
६० रेणवः १ लवः	१/८०० सेकण्ड
६० लवाः १ लीक्षकम्	१/१५ सेकण्ड
६० लीक्षकानि १ प्राणः	४ सेकण्ड

मूर्ताः स्थूलकालास्तु

६ प्राणः १ विनाडिका	२४ सेकण्ड
६० विनाडिका १ नाडिका	२४ मिनिट
सार्धिद्विनाडिका	१ हवर
३० नाडिका १ अहः ३० नाडिका रात्रिः अहोरात्रम् एकः दिनम्	
७ दिनानि	१ सप्ताहः
२ सप्ताहः ससन्धिः	१ पक्षः
२ पक्षः	एकः मासः
२ मासः	१ ऋतुः
३ ऋतवः	१ अयनम्
२ अयनम् १ मनुष्यसंवत्सरः एकं देवदिनम्	
३६० मनुष्यसंवत्सराः एकः देवसंवत्सरः	
४८०० देववर्षाणि	कृतयुगम्

३८०० देववर्षाणि	द्वापरयुगम्
२४०० दवेवर्षाणि	त्रेतायुगम्
१२०० देववर्षाणि	कलियुगम्
१२००० देववर्षाणि	१ महायुगम्
७१ महायुगानि	१ मन्वन्तरम्

१४ मनवः ९९४ महायुगानि ६ महायुगानि सन्धिः

१००० महायुगानि एकः ब्रह्मकल्पः एकं ब्राह्मं दिनम्

ब्रह्मणः परमायुः शतमिति सूर्यसिद्धान्ते।

अर्धांशः गतः। वयं ब्रह्मणः द्वितीये परार्धे अष्टाविंशतितमे कलौ युगे वैवस्वते मन्वन्तरे जीवामः इति गणना वर्तते। इदानीन्तन कालव्यवहारसंज्ञा हवर इत्याख्यं आङ्गलपदं अतिप्रसिद्धं वर्तते। तदस्माकं संस्कृते विद्यमानहोरापदादेवागतमस्ति।

सार्धद्वयनाडिका एका होरा भवति। होरा नाम एक हवर। अहोरत्रे एव एषा होरा अन्तर्भवति इत्यतः अहोरात्रशब्दात् होराशब्दस्य निष्पत्तिं कांक्षमानाः दैवविदः पूर्वापर–वर्णलोपेन होराशब्दं निष्पादयन्ति। तदुख्नं मिहिरेण – होरेत्यहोरात्रविलपमेके वाञ्छन्ति इति।

त्रयः मासविशेषा लोकव्यवहारे प्रसिद्धाः सन्ति। ते सावनमासः सौरमासः चन्द्रमासः इति।

सावनमासः

उदयादुदयान्तं हि सावनं दिनमिष्यते। तद्वत् इनोदयद्वयान्तरं तदर्क सावनं दिनम् इति सावनदिननिर्वचनं दृश्यते। सुर्योदयात् सूर्योदयपर्यन्तं सावनं दिनं भवति। त्रिंशत् सावनदिनानि एकः सावनमासः। सावनमासस्य संज्ञा लोके न दृश्यते।

सौरमासः

षष्ट्यधिक त्रिंशत् भागात्मके भगणचक्रे सूर्यस्य एकभागाभोगकालः एकं सौरं दिनं भवति। त्रिंशत् भागापूर्तिकाले एकराशिपूर्तिः भवति तदेव सौरमासः। सुर्यसंक्रान्तिद्वयान्तर्वर्तिकालः सौरमासः। सौरमासाः राशिनाम्नि प्रसिद्धाः मेषादयः द्वदशमासाः।

चान्द्रमासाः

सूर्याचन्द्रमसोः अन्तरं द्वादशभागात्मकः कालः तिथिः। एकं चन्द्रं दिनं भवति। तिथयः शुक्ले बहुले च पञ्चहदशदिनानि। आहत्य ३० दिनानि। अमान्तद्वयान्तर्वर्तिकालः एकः चान्द्रमासः। शुक्लप्रतिपदात् प्रारभ्यते। चान्द्रमासाः अपि द्वादश। चैत्रवैशाखादयः प्रसिद्धाः।

इति कालनिरूपणसंप्रदायविवरणं प्रोक्तं भवति।

25

Some Constructions in the Mānava Śulbasūtra

S.G. Dani

Abstract: The *Mānava Śulbasūtra*, while less sophisticated than the other Śulbasūtras, is seen to contain some mathematical ideas and constructions not found in the other Śulbasūtras. Here we discuss some of these constructions and discuss their significance in the overall context of the Śulbasūtra literature.

Introduction

AMONG the works from the Vedic period that have come down to us, the Śulbasūtras constitute a major source enabling understanding of that time concerning the mathematical aspects. Śulbasūtras were composed in aid of the activity around construction of *agni*s and *vedī*s (fireplaces and altars) for performance of the *yajña*s which, it is needless to add here, had a very important role in the life of the Vedic people. The Vedic community was fairly heterogeneous, though with a shared tradition and body of knowledge, and there would have been numerous Śulbasūtras, used by various local communities. Not surprisingly, very few have survived. Of the handful of extant Śulbasūtras, four are found to be significant from a mathematical point of view: The *Baudhāyana Śulbasūtra, Āpastamba Śulbasūtra, Mānava Śulbasūtra* and *Kātyāyana Śulbasūtra.*

While there is considerable uncertainty about the time when the Śulbasūtras were composed, it has now become customary among the commentators to assign to their composition the period 800–200 BCE, with the *Baudhāyana Śulbasūtra*, believed to be the earliest, to be from around 800–500 BCE. It is also concluded from various considerations that the *Mānava Śulbasūtra* is from a later period than the *Baudhāyana Śulbasūtra*, but is a little older than the *Āpastamba Śulbasūtra* and considerably so compared to the *Kātyāyana Śulbasūtra*; the ranges assigned typically are 650–300 BCE for the *Mānava* and *Āpastamba Śulbasūtra* and 400–200 for the *Kātyāyana Śulbasūtra*. Despite being the oldest the *Baudhāyana Śulbasūtra* is found to be better organized and more elegant in its presentation among all the four, while the *Mānava Śulbasūtra* is least appealing from these considerations.

It has also been the one to have received least attention in terms of editions, commentaries, etc. whether in traditional or in modern context, perhaps due to its lack of appeal. The first modern edition with English translation, due to Jeanette van Gelder (1963), is only a little over fifty years old, while for the others similar activity was undertaken well over 100 years ago, in the nineteenth and early twentieth centuries.

Notwithstanding its lack of appeal, there are some very interesting original observations in the *Mānava Śulbasūtra* in terms of the mathematical content, which in the overall context seem to have not received adequate attention. I may also put in a comment here that there seems to be a tendency among the scholars in the area to view the Śulbasūtras body of knowledge mostly as a totality and the special features of the individual Śulbasūtras are scarcely highlighted, except at a superficial level, while, on the other hand, there is no doubt that comparative studies between the individual Śulbasūtras could throw a good deal of light on various aspects of the Vedic civilization, especially as the Śulbasūtras are from different periods, and very likely also from different geographical regions of India. It is the aim of this article to highlight some of the unique features of *Mānava Śulbasūtra* compared to the other Śulbasūtras.

Circumferance of the Circle

During the ancient period, around the world the ratio of the circumference to the diameter of the circle was thought to be 3,[1] and the belief is also reflected in one of the *sūtra*s in the *Baudhāyana Śulbasūtra*; at one point there is an incidental reference to this, where a circular pit "with diameter 1 *pada* and circumference 3 *pada*s" is mentioned, indicating that the circumference was taken to be three times the diameter. The issue does not feature elsewhere in the *Baudhāyana Śulbasūtra* and in the *Āpastamba* and *Kātyāyana Śulbasūtra*. In the *Mānava Śulbasūtra*, however, one sees a recognition that the assumption is not correct. A verse in the *Mānava* (10.2.3.13 as per Kulkarni (1978) and 11.13 as per Sen and Bag (1983) numberings) states:

> *viṣkambhaḥ pañcabhāgaśca viṣkambhastriguṇaśca yaḥ* ।
> *sa maṇḍalaparikṣepo na vālamatiricyate* ॥

> A fifth of the diameter and thrice the diameter is the circumference of a circle, not a hair-breadth remains.

Viṣkambha, which also means supporting beam or bolt or bar of a door (see Monier-Williams and Apte), was the technical term used for the diameter of a circle. *Maṇḍala* stands for the circle and *parikṣepaḥ* is the term for the circumference. Even though the value described is considerably off the mark, the fact of recognition of the ratio being

---

[1] One may wonder why the value for the ratio was taken to be 3 across various cultures. My hypothesis on the issue is that the idea of the ratio being 3 dates back to the time when humans were yet to think in terms of fractions (except perhaps for "half", which may have meant a substantial portion that is not nearly the whole – as commonly used even now in informal conversations – rather than its precise value); it may be noted that while encounter with the circle, in the context of wheels, is at least over 5,000 years old, fractions seem to have appeared on the scene in a serious way, in Indian as well as Egyptian cultures, only around the first millennium BCE. The ratio is thus 3 in the sense that it is not 2 or 4, or even "three and half". The ingrained notion could have developed into a belief (often tagged also to religious authority). It was then not reconsidered for a long time, even after fractions became part of human thought process. The episodes such as discussed here mark a departure from the past.

strictly greater than 3 is worth taking note of, and so is the apparent exultation over the finding.[2]

It may be recalled here that in the Jaina tradition a similar recognition is seen in *sūryaprajñapti* (believed to be from fifth century BCE), where the classical value 3 for the ratio is recalled and discarded in favour of another value √10. The values could thus be contemporaneous, but evidently unrelated from a historical point of view, especially on account of the substantial difference in the values proposed, in numerical as well as structural terms.

A brief description of the location of the verse in the body of the *Mānava Śulbasūtra* would be in order here, to place the verse in context. Section 10.3 in which the verse occurs, at 10.3.2.13,[3] is the last of the three sections in the *Mānava Śulbasūtra*, referred by the *śulbakāra* as

[2] In van Gelder (1963) and in Kulkarni (1978) following it, the verse is wrongly interpreted as concerning determination of a square with the same area as the given circle: the translation of the verse is given as "Dividing the diameter of the circle into five parts and then individual parts into three parts each (thus dividing the diameter into 15 parts and taking away two parts) yields the side of a square with the same area as the circle. This is accurate to a hair-breadth." If the translation in the first part were to be correct then it would correspond to the formula for the side of the square with area equal to that of a given circle is 13 seen in the *Baudhāyana* (at 1.60) and also in the *Āpastamba* and *Kātyāyana Śulbasūtra*. The translation, however, is quite erroneous in many respects: occurrence of the word *viṣkambha* twice readily shows that it is not the individual parts that are being subdivided, and there is no reference at all to taking away two parts from the 15 subdivided parts. Besides, *parikṣepa* unambiguously corresponds to circumference, with the verb *parikṣip* meaning "to surround", "to encircle", etc. (see Monier-Williams, Apte), and not the area. It appears that having difficulties in interpreting the verse the translator chose to relate it to the 13/15 formula seen in the other Śulbasūtras. The translation in Sen and Bag (1983) on the other hand is along the lines described here.

[3] Actually the 10 is superfluous in these numbers, since the whole of *Mānava Śulbasūtra* is covered in sections of chapter 10; the numbering has to do with the translation of the *Mānava Śulbasūtra* in van Gelder (1963), in which the Śulbasūtra appears as chapter 10.

Vaiṣṇava; the significance of the name, and association with Viṣṇu, if any, is not clear from the contents of the section. The general narrative in the part containing the verse concerns description of construction of *vedī*s. Interestingly, after talking about the volume of the *vedī* called *śamitra vedī* the *sūtrakāra* states:

> *āyāmamāyāmaguṇam vistāraṁ vistareṇa tu* ।
> *samasyā vargamūlam yat tatkarṇaṁ tadvido viduḥ* ।
>
> Multiply the length by the length and the width by the width. It is known that adding them and taking the square root gives the hypotenuse.

The reader would recognize this statement as an equivalent form of what is called the Pythagoras Theorem, with the figure in question (not specified in the verse) being the rectangle.[4] It may also be noted that the statement is in quite a different form than in the other Śulbasūtras; in a way, while the other Śulbasūtras seem to be referring to geometric principle involved, considering in particular the areas of the squares over the respective sides, the exposition here is seen to be focused on *computing* the size of the hypotenuse from the sizes of the sides, without specific reference to the underlying geometry.

A few verses down from there, which concern practical details about the *vedī*s and the performance of *yajña*, we are led to another important mathematical statement, involving now the construction of a circle with the same area as a square.[5] In the Vedic literature

---

4 Kulkarni (1978) also mentions the right-angled triangle in this respect but there is no evidence, on the whole, of the Śulbasūtras discussing right-angled triangles.

5 It is argued in Hayashi (1990) that 10.3.2.10 gives rules both for squaring the circle, and circling the square, with the latter being the same as Baudhayāna's rule discussed earlier. The rule for the other direction, according to the interpretation in Hayashi (1990) is that given a circle, the perpendicular bisector of the equilateral triangle with the diameter of the circle as the side, is the side of a square with the same area as the circle. The argument involves an emendation of the extant text, which the author justifies also on considerations of grammar, but with →

this issue concerns constructing the *āhavanīya*, which is a square and *gārhapatya* which is circular with the same area;[6] along with there is also the semicircular figure with the same area to be constructed for the *dakṣiṇāgni*. The method described here for finding a circle with the same area as a given square is the same as given in the *Baudhāyana Śulbasūtra* in geometrical content, but formulated with a difference: in an isosceles triangle produced by the diagonals of the square, extend the perpendicular (as much as the semi-diagonal side of the triangle) and of the extra part of the semi-diagonal (beyond the side) adjoin a third part of it to the part within the square, to get the radius of the circle. As is well known (see in particular Dani (2010) for a discussion on this) this is not very accurate, but is interesting as an approximate construction. This is followed by two verses which concern doubling of area when measure of a side is replaced by that of the diagonal of the square. This is evidently related in this context with the construction of the *dakṣiṇāgni*, though it has not been explicitly mentioned, and has also not been brought out in the translations in Kulkarni (1978) and Sen and Bag (1983).

And then comes the cited verse for the circumference of the circle! What is the relevance that we can identify? We see that some circles have appeared on the scene, though what is involved about them are the *areas*. Nothing in the context warrants, apparently, consideration of the circumference. However, having got to the circles, seems to have inspired the author to mention, and that too with some gusto, something interesting that he had realized, namely, that the circumference is not just three times the diameter as people thought, but more than that, and one would have a safe estimate by adding

← it many aspects which are unclear from the earlier translations from van Gelder (1963) and Sen and Bag (1983) become clearer. As noted in Hayashi (1990) the above-mentioned rule for squaring the circle is unique to the *Mānava Śulbasūtra*. The rule however is not very accurate.

6 In Dani (2010) concerning the motivation for considering the problem of circling the square, I had made a reference to the *rathacakraciti*, which however seems to be an inadequate explanation – the primary motivation for the problem is very likely to have been the equality of the areas of *āhavanīya* and *gārhapatya*.

one-fifth of the diameter. Thus the statement (like much else actually, but it bears emphasis here) appears to be side input, from which it would be difficult to draw further inference about the thought process that may be involved.

Indeed, one may wonder why the *śulbakāra* chose the value 3⅕ for the correction, rather than something that would have been better, specifically like 1/6, if not 1/7. From the context, and the value itself, it is clearly an ad hoc value being adopted, essentially in the context of becoming aware of the classical value of 3 for the ratio is not satisfactory, and that something remains. I may reiterate here in this respect that the verse notes that "not a hair-breadth remains", which is what *atiricyate* corresponds to, with the verb *atiric* meaning "to be left with a surplus" (see Monier-Williams), and is strictly not a reference accuracy in terms of both lower and upper estimates (as treated, for instance, in Kulkarni (1978)). But having recognized that the value should be more than 3, why and how did 3⅕ come to be chosen for it. The value 3⅐, which would be appropriate in hindsight, would perhaps would have been rather odd (lacking in aesthetic appeal, which is often a consideration while making ad hoc choices) to think about at that time. However, why not, say 3 ⅙, which would have been much closer to the correct value? Thinking of a sixth would seem simpler and natural compared a fifth part, it being half of a third, and division into three parts is easier operationally, than into five parts, and then halving would of course be the trivial next step.

The *sūtrakāra*, however, prefers to consider division into five parts. A clue for this seems to lie in the decimal place value system of representation of numbers (writing numbers to base 10, as we do now). For a number written in this system, it is much easier to compute its fifth part than the third, or any other, part. Indeed, the *Mānava Śulbasūtra* shows preference to using decimally convenient divisions in other contexts as well. The verse following the cited one, for the circumference describes the size of a square inscribed in a circle, viz. with vertices on the circumference. It may be noted that the desired size would be $1/\sqrt{2}$ times the diameter of the circle. The prescription given is to divide the diameter into 10 parts and take away 3 parts; thus 7/10 is used as a (n approximate) value for $1/\sqrt{2}$. Actually, for $\sqrt{2}$, there was a standard approximate value 17/12 adopted in the Śulbasūtras, according to which the desired ratio would be 12 out of 17 parts, which would be more accurate, but Mānava adopts the proportion 7 out of 10, suggesting preference for decimal division. In the verse for a new construction for circling the square, which we shall discuss in the next section, there is a division into 5 parts involved. It may also be recalled here that the major large unit involved in the Śulbasūtras is *puruṣa* and there is a subunit *aratni*, which is a 5th part of *puruṣa*. This may also be looked upon as a factor, related to the use of the decimal system, which would have encouraged considering division into 5 parts. I may also recall here that in the construction of various *vedi*s that are described, division by 5 is involved in many computations.

We conclude this discussion with another small related observation. Granting that the value of the circumference to diameter ratio was recognized by the *śulbakāra* as being greater than 3, and that he looked for additional decimal parts after which "nothing will remain", ⅕ is the right choice; 1/10 would have been closer, but it is *less* than the correct value.

Circling the Square

As noted in the last section the problem of circling the square, namely, of describing a circle with the same area as a given square, had attained considerable importance in the Śulbasūtras period. It may be emphasized that the framework envisaged for the problem is quite different from the analogous problems in Greek mathematics, where the constructions were sought to be performed with *only* the ruler and compass, and any comparison of the achievements of the ancient Indians, in the context of the Greeks "not having been successful" with the problem are facile and irrelevant. The constructions given are important in terms of historical development of mathematical ideas and need to be viewed only as such.

We have gone over the geometric construction given in the *Baudhāyana Śulbasūtra* for drawing a circle with the same as a given square. As noted there, the result it produces is not very accurate, and in fact involves an error of the order of 1.7 per cent (see Dani (2010) for more details in this respect). In course of time, the suspicions would have gained weight, serving as motivation to look for an alternative construction, and one seems to find such an attempted construction in the *Mānava Śulbasūtra*, which we shall now discuss.

The construction in question is described in a verse which follows right after the contents discussed in the last section here (in 10.3.2.15 as per the numbering of Kulkarni (1978) and 11.15 of Sen and Bag (1983). The verse is:

> *caturasraṁ navadhā kuryāt dhanuḥ koṭistridhātridhā* ।
> *utsedhātpañcamaṁ lumpetpurīṣeneḥ tāvatsamaṁ* ॥

The first part of the verse may be translated, quite unambiguously, as:

> Divide the square into nine parts, (by) dividing the horizontal and vertical sides into three parts each.

Unfortunately, arriving at the right translation of the rather terse second part of the verse, and its interpretation, call for additional inputs of contextual nature, and want of these seems to have confused earlier translators: in Sen and Bag (1983) the authors translate the second part as:

> drop out the fifth portion (in the centre) and fill it up with loose earth.

And in the commentary section, they comment:

> Possibly these are not problems of quadrature of the circle. Ordinary squares are drawn without any mathematical significance.

The comment seems quite unwarranted, though it may be emphasized there that nothing in the verse specifically indicates that it does concern a quadrature formula, or procedure towards one. In Kulkarni (1978), following van Gelder (1963), the second part is interpreted (in Marathi and Hindi equivalents) as:

> from the part jutting out take away one-fifth part and draw a circle with the remaining part as the radius.

Here the word *utsedha* is interpreted to mean the part of the trisectors (arrived at in the first part) that is jutting out on either side of the square, until meeting the circle passing through the vertices of the square. One-fifth of that is subtracted from the segment of the trisector up to its midpoint and the remaining part is taken as the radius of the prescribed circle. Implementing the procedure accordingly, they calculate the radius; it, however, turns out to be much too large for the corresponding circle to have the same area as the square, thus putting the interpretation into question, but the matter is left at that, with no comment.

Another interpretation of the verse was given by R.C. Gupta (1988) (see also Gupta (2004)). Here *utsedha* is interpreted to mean "height" and is associated with the "height", viz. the radius, of the semi-circle from the circle through the vertices of the square. Thus, the author infers that the radius of the prescribed circle is meant to be $4/5^{\text{th}}$ of the circumscribing circle. With this interpretation the area of the circle produced, starting with the unit square, works out to be $8\pi/25$, and thus the procedure corresponds to a value of π as 25/8. This is a good value by the Śulbasūtras standards. However, the interpretation is unsatisfactory in various ways. First and foremost, the interpretation does not involve the first part of the verse at all. It is inconceivable that the *śulbakāra* first asks you to elaborately divide the square into nine parts, and in the following line gives a procedure for the quadrature problem which has nothing to do with the subdivision. Second, in the second part if it was just the radius of the circumscribing circle to be used as a reference, why would it be referred to with the unusual word *utsedha*, which does not occur anywhere else in the *Mānava* (or in other Śulbasūtras), rather than in terms of the diameter of the circle, which is something that occurs so frequently in Śulbasūtra geometry.

For a faithful interpretation of the verse it seems imperative that it must involve the trisectors of square introduced in the first part; also *utsedha* must have something to do with the trisectors and the choice of the unusual term must have to do with that the trisectors also do not occur anywhere else. Thus, it would seem that the interpretation in van Gelder (1963) and Kulkarni (1978) is on the right track inasmuch as it focuses on considering individually the lines trisecting the given square along each of the sides, extended up to the circle passing through the vertices of the circle. The circle is indeed being described in terms of certain points on these lines. The main difficulty however seems to be in understanding which points are meant. Evidently, the interpretation with regard to the points, and how they are to be used (see more on this below), adopted in van Gelder (1963) and Kulkarni (1978) does not seem to the right one, as it is way off the mark.

The overall formulations and symmetry considerations suggest that we are to pick two points on each of the four lines that trisect the square along a side, located symmetrically (and hence at the same distance from the centre of the square) and the circle through these points is the desired circle; this in a way explains the explication through "covering with loose earth", as the totality of the eight points is indicative of a circle which is what is to be covered. Now, which are the two points on each of the lines? One would be in a better position to figure out what the *śulbakāra*'s line of thought, if one keeps in mind the Baudhāyana construction of the circle, described earlier. Recall that there the bisector of the square is extended until meeting the circle through the vertices, and $1/3^{rd}$ of the part is added to the segment within, to get the radius of the circle that is sought after; one can alternatively think of this as identifying the point through which the circle should pass (the centre of the circle is of course understood to be the centre of the square). The new idea now is that instead of the bisectors of the squares we are considering the trisectors. On the bisectors the points in question were chosen to be at $1/3^{rd}$ of the jutting out part, *from the side of the square*. One now needs to look for a similar number, and the analogous point on the trisectors, to complete the analogous construction. The number is picked to be $1/5^{th}$; the choice could have been based on intuition, and the point is now meant to be on the trisector at $1/5^{th}$ of the jutting out part, from the side of the square. At this point the analogy with the Baudhāyana construction throws open, to our minds, two possibilities, one is to take the segment of the trisector up to the point thus constructed either from the midpoint of the trisector, or the centre of the square; in the case of the Baudhāyana construction with the bisector the two coincide, but here they are different. For some reason in Kulkarni (1978) the former interpretation is favoured (with respect to the point picked there, on which was commented upon above). However, viewed in the full context, it is the other interpretation that may be seen to be more appropriate. The *śulbakāra*s do not in general try describe a number for the radius, but a region to be covered determined by some point (or a collection of points), and second, in the overall context of the description of the construction the midpoint of the trisector has no relevance (and has not been referred to). Once these points are noted, the inference would be that the prescribed circle passing through the point(s) as above on the trisectors, at one-fifth of the jutting out part from the side of the square. One may now rewrite the interpretation of the second part, referring to the collection of the 8 points, as:

> on the parts jutting out mark the points at one-fifth (from the square) and draw the circle through them.

A simple calculation shows that for a square with unit side length the radius of the circle is

$$\frac{1}{2}\left\{1+\frac{1}{5}\left(\frac{\sqrt{17}}{3}-1\right)\right\}^2+\frac{1}{8},$$

and this yields the area of the circle to be 0.994 ..., a much more accurate value compared to the earlier one, with an error of only about 0.5 per cent, in place of 1.7 per cent (see Dani 2010) for details of the calculations and other related comments). Thus, from a mathematical point of view, this turns out to be a good choice. We see also that it emerges naturally as a generalization of the Baudhāyana construction in terms of development of ideas. As a result, it seems reasonable to expect that this is what the *śulbakāra* had in mind. The interpretation incorporates all the components of the verse, and all the ingredients needed in the formulation may be seen to be present in the verse, in their natural order. The author is hopeful that the interpretation would be confirmed to be valid by expert Sanskritists, from a linguistic point of view, possibly after some emendation that could be justified based on considerations of corruption on account of one or other factors.

References

Amma, T.A. Saraswati, 1979, *Geometry in Ancient and Medieval India*, Delhi: Motilal Banarsidass (repr. edn 2000).

Apte, Vaman Shivram, 1970, *The Students' Sanskrit English Dictionary*, Delhi: Motilal Banarsidass.

Dani, S.G., 2010, "Geometry in the Sulbasutras", in *Studies in History of Mathematics*, Proceedings of Chennai Seminar, ed. C.S. Seshadri, New Delhi: Hindustan Book Agency.

Datta, Bibhutibhusan, 1932, *Ancient Hindu Geometry: The Science of the Śulba*, Calcutta: Calcutta University Press (repr., New Delhi: Cosmo Publications, 1993).

Gupta, R.C., 1988, "New Indian Values of π from the Mānava Śulvasūtra", *Centaurus*, 31: 114-25.

———, 2004, "Vedic Circle-square Conversions: New Texts and Rules", *Ganita Bharati*, 26: 27-39.

Hayashi, Takao, 1990, "A New Indian Rule for the Squaring of a Circle: Mānava Śulbasūtra 3.2.9-10", *Ganita Bharati*, 12: 75-82.

Kulkarni, Raghunath P., 1978, *Char Shulvasūtre* (in Marathi), Mumbai: Maharashtra Rajya Sahitya Sanskrit Mandal, Hindi translation *Char Shulbsūtra*, Ujjain: Maharshi Sandipani Rashtriya Vedavidya Pratishthana, 2000.

Monnier-Williams, 2002, *A Sanskrit English Dictionary*, (corrected edition) Delhi: Motilal Banarsidass (originally published in 1899).

Plofker, Kim, 2008, *Mathematics in India: 500 BCE – 1800 CE*, Princeton NJ: Princeton University Press.

Sen, S.N. and A.K. Bag, 1983, *The Śulbasūtras*, New Delhi: Indian National Science Academy.

van Gelder, Jeanette, 1963, *The Mānava Śrautasūtra*, English translation, New Delhi: Indian National Science Academy.

26

Geometry in Śulbasūtras

Sudhakar C. Agarkar

Abstract: India has a long tradition of mathematics. A variety of mathematical principles were used in rituals followed in ancient Indian society. Although most of these principles were passed on from one generation to another orally, some of them have been recorded in *sūtra* forms. Śulbasūtras composed sometimes in 800 BCE is one such document. It depicts some of the major theorems of modern geometry. Pythogoras theorem can be cited as an example. The *Bodhāyana Śulbasūtra* clearly states the theorem in the context of a diagonal of a rectangle. It goes on describing how to draw figures like square, rectangle and circle. Śulbasūtra also describes the methods of transformation of figures. Procedures for transforming circle into a square, a square into a circle, circle into a rectangle and a rectangle into a circle are given clearly. Study of these procedures and principles brings out clearly how deep geometrical concepts were embedded into the thinking of our ancestors. This paper attempts to highlight geometrical knowledge of ancient Indian mathematicians as presented in Śulbasūtras.

Keywords: Ancient geometrical knowledge, mathematics in rituals, Śulbasūtras, transformation of figures.

Introduction

Our search for ancient mathematical literature takes us to Śulbasūtras that deal with the rules for the measurement and construction of various sacrificial fire places (*agni*) and altars (*vedī*s). Śulbasūtras do not describe geometry in the forms of formulae or statements of theorems. Instead, they give guidelines for the accurate layout of altars and fire places. In spite of the above limitations the Śulbasūtras have a special place in the history of Indian mathematics.

The name Śulbasūtras is derived from two Sanskrit words *śulba* and *sūtra*. *Śulba* in Sanskrit literally means a cord, a rope or a string. It is derived from the basic word *sulb* or *sulv* meaning to mete out or to measure. The word *sūtra* means aphorism or a short rule. In ancient India, there was a practice of using *śloka*s for writing. These *śloka*s give a lot of meaning in shortest possible words. Since most of the knowledge in ancient India was passed on from one generation to another through oral mode, this *sūtra* mode of writing helped them to remember and reproduce the matter correctly.

There are many versions of Śulbasūtras available. Out of these the Śulbasūtras of Baudhāyana, Āpastamba, Kātyāyana and Mānava are well known.

- *Baudhāyana Śulbasūtra*: 323 *sūtra*s in 21 chapters
- *Āpastamba Śulbasūtra*: 202 verses in 21 chapters
- *Kātyāyana Śulbasūtra*: 61 verses in 6 chapters
- *Mānava Śulbasūtra*: 233 verses in 16 chapters

Exact time of the composition of these treatises is not known. But historians give the following chronology:

- *Baudhāyana Śulbasūtra*: 800–500 BCE
- *Mānava Śulbasutra*: 750–690 BCE
- *Āpasatamba Śulbasūtra*: 650–450 BCE
- *Kātyāyana Śulbasūtra*: 400–300 BCE

Śulbasūtras provide useful information about geometrical figures and their transformations. As T.A. Saraswathi Amma (2017) mentions in her book, *Geometry in Ancient and Medieval India*, the geometrical contents of the Śulbasūtras can be broadly divided into three categories: 1. Theorems expressly stated, 2. Constructions, and 3. Geometrical truths implied in constructions.

Theorem of Square of Diagonal

The theorem popularly known as Pythagoras Theorem is mentioned in and the *Baudhāyana Śulbasūtra* in the following *śloka*:

दीर्घचतुरस्रस्याक्ष्णयारज्जुः पार्श्वमानीतिर्यंगमानी।
चयतपृथग्भुतेकुरूतस्तदुभयंकरोति।।

The diagonal cord of a triangle makes both (squares) that the vertical side and the horizontal side make separately. Pictorially the theorem is shown in *fig*. 26.1.

The theorem is also extended to square. In this case since the vertical and horizontal sides are same it is stated "The diagonal cord of a square makes double the area" in the following *śloka*:

चतुरस्रस्याक्ष्णयारज्जुर्द्विस्तावतीभूमिकरोति।

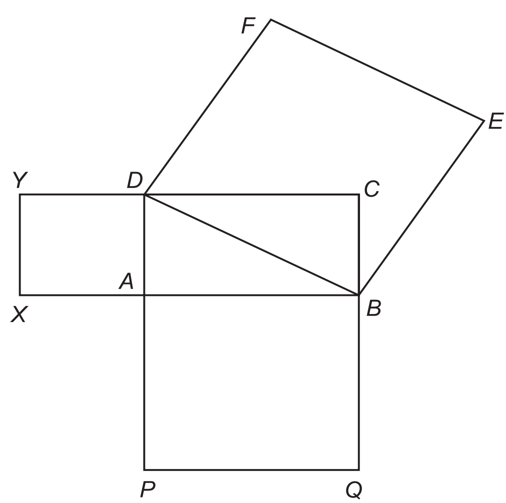


fig. 26.1 The Kātyāyana form of Pythagoras's Theorem

Perpendicular Bisector of a Line

*Śulbakāra*s had the duty to find out the directions. Using the shadow of the sun they could determine the east–west direction. In order to find the north–south direction they used to draw a perpendicular bisector to the east–west line. This procedure is described in the following *śloka*:

> तदंतरंरज्वाभ्यस्य,पाशोकृत्वा, शङ्क्वोपाशोप्रतिमुच्य।
> दक्षिणायम्यमध्येशंकुनिहंतिएवमुत्तरतः सोदीची।।

> Doubling the distance between the end points on a cord and making ties one fixes the ties on the pins, stretches the cord to the south and strikes a pin at the middle point. Similarly to the north. That is the north–south line, a perpendicular bisector of an east–west line.

Actual procedure of obtaining the perpendicular bisector is shown in *fig*. 26.1. This method looks similar to the modern method of obtaining the perpendicular bisector where arcs are drawn instead of stretching the ropes.

Construction of a Square

Square is a common figure in geometry. *Śulbakāra*s suggested a practical method of obtaining the squares. The most primitive method of getting a square is based on drawing a perpendicular bisector to a given line from its midpoint. It suggests to take a bamboo equal to the length of the side of a square. It should have holes at the ends and at the middle. Place the bamboos at the right angles to the first one. Slip the middle hole of the bamboo so that

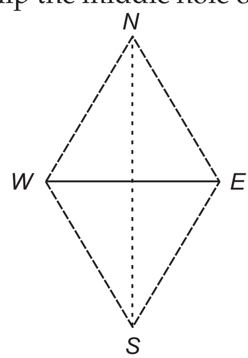


fig. 26.2: Procedure of obtaining the perpendicular bisector

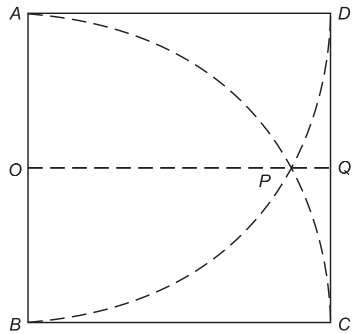


fig. 26.3: How to make a square: a practical method

its ends touch the arcs. The ends of the bamboo mark the corners of the square. Since the bamboo is tangential to the arc it makes a right angle with the other bamboo touching it (*fig*. 26.3). Let *AB* be the line with *O* as its centre. A bamboo equal to the length of *AB* is first pivoted at *A* and the free end is rotated as shown. Then the bamboo is pivoted at *B* and the other end is rotated. These two arcs meet at *P*. Join *OP* and extend. Finally place the bamboos or draw lines tangential to these curves. We thus get square *ABCD*.

Square Equal to Sum of Two Squares

The *Āpastamba Śulbasūtra* suggests a very simple method of getting a square equal to two squares in the *śloka* as given below (see *fig*. 26.4):

> ह्सीयसः करण्या वर्षीयसो वृध्दमुल्लिखेत।
> वृध्द्रस्याक्ष्णयारज्जुरुभे समस्यति।।

> With the side of a smaller one a segment of the bigger one should be cut off. The diagonal cord of the segment will combine the two squares.

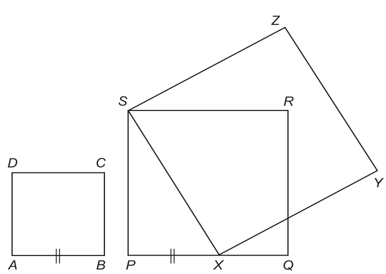


fig. 26.4: Method to find a square equal to the sum of two given squares is given in all Śulbasūtras

Let *ABCD* and *PQRS* be two squares. Mark the point *X* taking the distance equal to the side of a smaller square. Join *X* with the vertex *S*. The length of this line should be the length of the new square. Thus, square *XYZS* is equal to two squares *ABCD* and *PQRS*.

Square Equal to Difference of Two Squares

Even the method of finding out a square equal to the difference of two squares in the *Āpastamba Śulbasūtra* is quite simple. It states:

> चतुरश्राच्चतुरश्र निर्जिहीर्षन यावन्निर्जिहीर्षेत तस्य करण्या वृध्दमुल्लिखेत।
> वृध्दस्य पाश्र्वमानी अक्ष्णया इतरत पाश्र्व उपसंहरेत सा यत्र निपतेत्तदपछिंध्यात।।

> Wishing to deduct a square from a square, one should cut off a segment by the side of the square to be removed. One of the lateral sides is drawn diagonally across to touch the other lateral side. The portion of the side beyond this point be cut off.

The procedure is illustrated in *fig*. 26.5. Let *ABCD* be the larger square and *AE* be the side of the smaller square. Mark a point *E* equal to the length of small square side. Draw *AD* diagonally until it touches *EF* at *P*. *EP* will be the side of square after subtraction.

Converting Rectangle into a Square

As stated above the Śulbasūtras give procedures for transformation of figures. It would be appropriate to see some of them. As a first case let us take the conversion of a rectangle into a square. The *Āpastamba Śulbasūtra* gives the following procedure for this conversion:

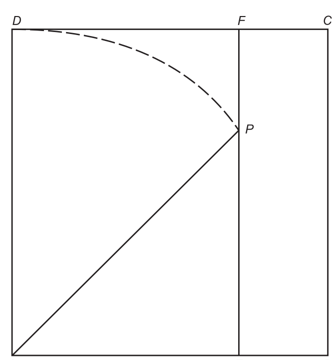


fig. 26.5: Method to find out a square equal to the difference of two squares

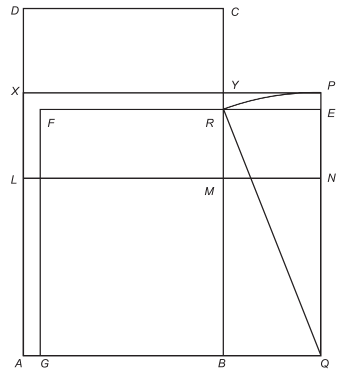


fig. 26.6: Finding a square equal to a given rectangle

दीर्घचतुरश्रं समचतुरश्रं चिकिर्षण तिर्यक मान्यापच्छिद्य।
शेषं विभज्योमयत उपदध्यात, खण्ड मागंतुना संपूरयेत।।

Wishing to turn a rectangle into a square, one should cut off a part equal to the transverse side and the remainder should be divided into two and juxtaposed at two sides. The bit at the corner should be filled in by an imported bit.

The procedure to be followed is described below along with (see *fig*. 26.6). The rectangle *ABCD* is given. Let *L* be marked on *AD* so that *AL* = *AB*. Then complete the square *ABML*. Now bisect *LD* at *X* and divide the rectangle *LMCD* into two equal rectangles with the line *XY*. Now move the rectangle *XYCD* to the position *MBQN*. Complete the square *AQPX*. Now rotate *PQ* about *Q* so that it touches *BY* at *R*. Then *QP* = *QR* and we see that this is an ideal "rope" construction. Now draw *RE* parallel to *YP* and complete the square *QEFG*. This is the required square equal to the given rectangle *ABCD*.

Converting a Square into a Rectangle

The procedure to convert a square into a rectangle as given in the *Āpastamba Śulbasūtra*:

यावदिच्छं पार्श्वमान्यौ वर्धयित्वा उत्तरपूर्वा कर्णरज्जुमायच्छेत।
सा दीर्घचतुरश्रमध्यस्थायां समचतुरश्रतिर्यंमान्यां यत्र निपतति
तत उत्तर दक्षिणाशं तिर्यंग्मानी कुर्यात, तद दीर्घचतुरश्रं भवति।।

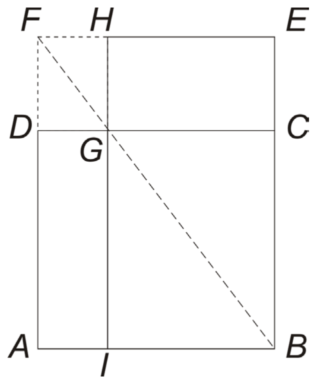


fig. 26.7: Procedure to convert a square into a rectangle

> Producing the sides of square eastward to the desired length of a lateral side one should draw the north eastern diagonal. The part of the transverse side to the north of the point where the diagonal cuts it is to be discarded and its southern part is to be made the transverse side of the rectangle.

The procedure is illustrated in *fig*. 26.7. Let *ABCD* be given the square we wish to convert into a rectangle. Produce *AD* and *BC* to *F* and *E* so that $AF = BE =$ the required side of the rectangle. Complete the rectangle *ABEF* and join the diagonal *BF* cutting *CD* in *G*. Through *G* draw a straight line *IH* parallel to the side of the square. *IBEH* is the required rectangle equal to the square *ABCD*.

Converting a Square into a Circle

Śulbasūtras also give guidelines to convert a square into a circle. An attempt is made to describe this procedure with illustration (*fig*. 26.8). Let *ABCD* be the given square. First find the centre of this square, let it be *O*. Connect *O* with the midpoint of *DC* (*P*) and extend the line. Now rotate *OD* to get the point *E*. Obtain *Q* on *PE* such that *PQ* is one third of *PE*. The required circle has centre *O* and radius *OQ*.

Converting a Circle into a Square

All the Śulbasūtras contain a method to square the circle. It is an approximate method based on constructing a square of side $^{13}/_{15}$ times the diameter of the given circle. In *fig*. 26.9 *XY* is taken as

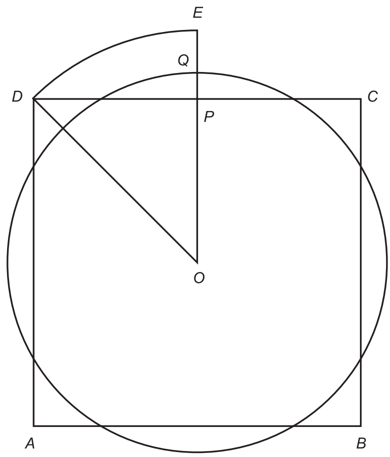


fig. 26.8: The Śulbasūtras method of "circling the square"

$^{13}/_{15}$ part of the diameter *AB*. The circle passing through *X* and *Y* is the required circle equal to the given square. In this case $\pi = 4 \times (^{13}/_{15})^2 = {}^{676}/_{225} = 3.00444$. So it is not a very good approximation. None the less, a circle closely equal to the area of the square can be obtained whenever required.

Geometrical Truths Implied

Even though, not explicitly stated, there are many geometrical truths that are implied in the construction procedure suggested in Śulbasūtras. Some of them are mentioned here:

1. The circle is a locus of points at constant distance from a given point.

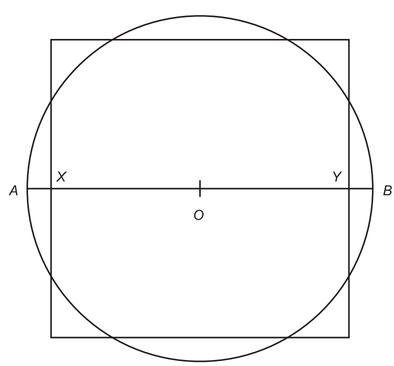


fig. 26.9: The Śulbasūtras $^{13}/_{15}$ method of "circling the square"

2. The perpendicular bisector is the a locus of points at a constant distance from a given point.
3. The tangent to a circle is perpendicular to the radius at the point of contact.
4. A finite straight line can be divided into any number of equal parts.
5. The diagonal of a rectangle or a square bisects them.
6. The figure joined by the midpoints of the adjacent sides of a square is itself a square.

Conclusions and Implications

Śulbasūtras are important treatises of ancient Indian mathematics. Although written to construct Vedic altars, they possess important geometrical information (Thakura Feru 1987). The techniques suggested in Śulbasūtras are useful even in modern days. Hence, school and college students in India must be made familiar with this literature so that they get the glimpse of wisdom possessed by our forefathers.

Engineering needs a lot of geometry. They should be made to follow the procedure described in these treatises. It will facilitate the ability to construct different geometrical figures. Carpenters and plaster workers are seen using these techniques quite often with the help of a rope. The ancient mathematics behind the procedure followed must be clarified to them (Kulkarni 1998).

References

Kulkarni, R.P., 1998, *Engineering Geometry of Yajña, Kuṇḍas and Yajñamaṇḍapas*, Pune: Jñāna Prabodhinī.

Saraswathi Amma, T.A., 2017, *Geometry in Ancient and Medieval India*, 2nd repr., Delhi: Motilal Banarsidass.

Thakura Feru, A.D., 1987, *Vastusāra Prakaraṇam*, English tr. R.P. Kulkarni, Pune: Jñāna Prabodhinī.

27

Development of Geometry in Ancient and Medieval Cultures

Shrenik Bandi

Abstract: The important branch of mathematics which received earliest attention was geometry and it is well explained in the texts of ancient and medieval cultures. In this paper I make an attempt to explore how geometry was developed and also discuss various results obtained by Vedic and Jaina scholars.

The beginning of geometry can be traced to ancient Mesopotamia, Egypt, Babylonia and India. Geometrical concepts were used in the development of towns in Indus Valley Civilization. We find many geometrical patterns in nature. Pythagoras was probably one of the first to give a deductive proof of Pythagoras Theorem. Thales expanded the range of geometry.

Geometry flourished in India, Arabia, China and Europe in ninth century. Analytic geometry, projective geometry, non-Euclidean geometry and so on were developed in seventeenth century.

Results related to rational right-angle triangle and conversion of one figure into other, all are mentioned in the Śulbasūtras. Louis Renou lists eight Śulbasūtras of which most notable are the Baudhāyana, Āpastamba and Kātyāyana. In the construction of *mahāvedī* and altars properties of right-angle triangle were used. M. Cantor and others recognize that Pythagoras Theorem was

known to Indians before eighth century BCE. The method for finding the area of a triangle $\Delta = \sqrt{s(s-a)(s-b)(s-c)(s-d)}$ was known. Derivation of relation $\frac{abc}{4r} = \Delta$ is given in the Vedic text.

The Jaina texts – the *Bhagavatīsūtra*, *Tattvārthādhigamasūtra Bhāṣya*, *Jambūdvīpasamāsa*, *Tiloyapaṇṇattī*, *Bṛhatkṣetrasamāsa*, *Laghukṣetrasamāsa*, *Jambūdvīpapaṇṇatti Saṅgaho* and *Trilokasāra* contain detailed knowledge of geometry. I have illustrated and derived some of the results from Jaina texts. The epithet is *kṣetra-gaṇita*, *rekhā-gaṇita* and *kṣetramiti*. In the *Sūryaprajñapti* ellipse was known by *viṣamacakravāla*. Perimeter of the ellipse $P \approx \sqrt{4a^2 + 6b^2}$ and area of the ellipse $\approx P \times \frac{2b}{4}$ were given. Now geometry is applied to computer science, crystallography and number theory.

Keywords: Euclidean, lines, triangle, quadrilateral, circle, Pythagorean, Śulbasūtras, Vedas, *kṣetra-gaṇita.*

Introduction

Geometry (from the Ancient Greek: γεωμετρίία; *geo* – "earth", *metron* "measurement") is a branch of mathematics concerned with shape, size, relative position of figures and the properties of space. It arose independently in a number of early cultures and it was a collection of empirically discovered principles. The earliest recorded beginnings of geometry can be traced to ancient Mesopotamia and Egypt in the second millennium BCE (Friberg 1981; Neugebauer 1969). By the third-century BCE, geometry was put into an axiomatic form by Euclid, whose treatment, Euclid's elements set a standard for many centuries to follow (Turner et al. 1998: 1).

Greek expanded the range of geometry to many new kinds of figures, curves, surfaces and solids; they changed its methodology from trial and error to logical deduction. Geometry began to see elements of formal mathematical science emerging in the West as early as the sixth century BCE (Boyer 1991: 43). Euclidean geometry includes the study of points, lines, planes, angles, triangles, congruence, similarity, solid figures, circles and analytic geometry (Schmidt et al. 2002). Topology is the field concerned with the properties of geometric objects that are unchanged by continuous mappings. Convex geometry investigates convex shapes in the Euclidean space and its more abstract analogues, Algebraic geometry studies geometry through the use of multivariate polynomials and other algebraic techniques. Discrete geometry is concerned mainly with questions of relative position of simple geometric objects, such as points, lines and circles.

Geometrical Pattern Found in Nature

Living things like orchids, hummingbirds and the peacock's tail have abstract designs with a pattern and colour that artists struggle to match (Forbes 2012). Mathematics seeks to discover and explain abstract patterns or regularities of all kinds (Steen 1998).

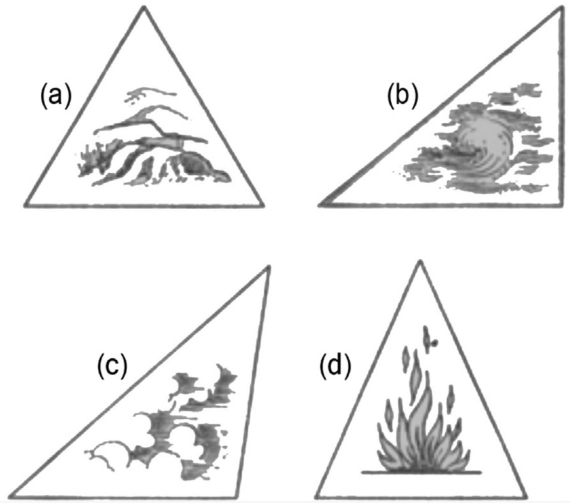
Symmetry

Symmetry is universal in living things. Animals mainly have bilateral or mirror symmetry, as do the leaves of plants and some flowers such as orchids (Stewart 2001: 48-49). Plants often have radial or rotational symmetry, as do many flowers and some groups of animals such as sea anemones. Fivefold symmetry is found in the echinoderms, the group that includes starfish, sea urchins and sea lilies.

Fivefold symmetry: Starfish

Rotational symmetry: *Cycas circinalis*

Example of different shapes of triangle compare with the nature:



Figures of triangle from the nature.

(a) An equilateral triangle (i.e. one of which all three sides are equal) is the elemental earth form; (b) a right-angled triangle is the spirit of water (to find spirit of water is the most advanced kind of magic); (c) a scalene triangle with no equal sides is the spirit of the air; and (d) an isosceles triangle (i.e. one of which only two sides are equal) is the elemental fire

Geometry in Early Period

The earliest known unambiguous examples of written records, from Egypt and Mesopotamia dating about 3100 BCE, demonstrate that ancient peoples had already begun to devise mathematical rules and techniques useful for surveying land areas, constructing buildings and measuring storage containers. The earliest recorded beginnings of geometry can be traced to the early people of the ancient Indus Valley Civilization and ancient Babylonian civilization from around 3000 BCE. There were some surprisingly sophisticated principles, and it might be hard put to derive some of them without the use of calculus; the Egyptians had a correct formula for the volume of a frustum of a square pyramid of Indus Valley Civilization.

Development of Geometry in Different Countries since Ancient Time

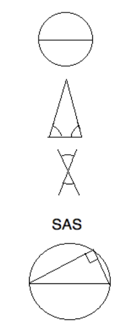
BABYLONIAN GEOMETRY

There have been recent discoveries showing that ancient Babylonians might have discovered geometry nearly 1,400 years before the Europeans. The Pythagorean Theorem was also known to the Babylonians.

EARLY GREEK GEOMETRY

The early history of Greek geometry is unclear, because no original sources of information remain and all of our knowledge is from secondary sources written many years after the early period.

Thales (635–543 BCE) of Miletus (now in south-western Turkey), used geometry to solve problems such as calculating the height of pyramids and the distance of ships from the shore. He is credited with the first use of deductive reasoning applied to geometry, by deriving four corollaries to Thales' Theorem (Boyer 1991: 43). Thales strongly believed that reasoning should supersede experimentation and intuition, and began to look for solid principles upon which he could build theorems. This introduced the idea of proof into geometry and he proposed some axioms that he believed to be mathematical truths.



Thales' Theorems

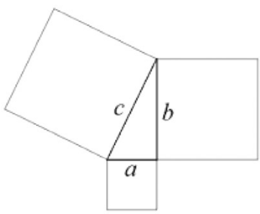
- A circle is bisected by any of its diameters.
- The base angles of an isosceles triangle are equal.
- When two straight lines cross, the opposing angles are equal.
- An angle drawn in a semi-circle is a right angle.
- Two triangles with one equal side and two equal angles are congruent.

It is unclear exactly how Thales decided that the above axioms were irrefutable proofs, but they were incorporated into Greek mathematics and the influence of Thales would influence countless generations of mathematicians.

CLASSICAL GREEK GEOMETRY

In ancient Greek, geometry was the crown jewel of their sciences, reaching a completeness and perfection of methodology that no other branch of their knowledge had attained. They recognized that geometry studies "eternal forms", of which physical objects are only approximations; and developed the idea of the "axiomatic method", which is still in use.

Pythagoras (582–496 BCE), of Ionia and later Italy, then colonized by Greeks, may have been a student of Thales. The theorem that bears his name may not have been his discovery, but he was probably one of the first to give a deductive proof of it. Pythagoras established the Pythagoreas School (Eves 1990).



The Pythagoreans added a few new axioms to the store of geometrical knowledge:

- The sum of the internal angles of a triangle equals two right angles (180°).
- The sum of the external angles of a triangle equals four right angles (360°).
- The sum of the interior angles of any polygon equals $(2n-4)$ right angles, where n is the number of sides.
- The sum of the exterior angles of a polygon equals four right angles, however of many sides.
- For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Hippocrates took the development of geometry further. He was the first to start using geometrical techniques in other areas of mathematics. He studied the problem of squaring the circle which is not perfect, simply because pi (π) is an irrational number.

APOLLONIUS OF PERGA (262–190 BCE)

He was a mathematician and astronomer, and he wrote a treatise called *Conic Sections*. He is credited with inventing the words ellipse, parabola and hyperbola, and is often referred to as the great Geometer.

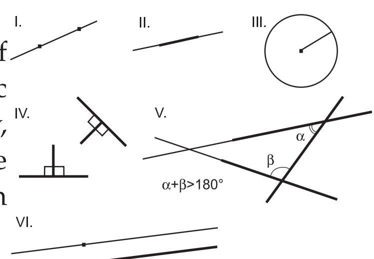
GREEK GEOMETRY AND ITS INFLUENCE

Greek geometry eventually passed into the hands of the Islamic scholars, who translated it and added to it. In this study of Greek geometry, there were many more Greek mathematicians and geometers who contributed to the history of geometry.

EGYPT GEOMETRY (300 BCE))


Euclid

Euclid was associated with the cosmopolitan University of Alexandria. He may well have been an Egyptian or a Jew (Hogben 1967: 118), but like others of the school he wrote in Greek his thirteen books composed about 300 BCE, Euclid himself wrote eight more advanced books on geometry. He was brought to the university at Alexandria by Ptolemy I, King of Egypt. Around 300 BCE, geometry was revolutionized by Euclid, whose Elements, widely considered the most successful and influential textbook of all time (Boyer 1991: 119), introduced the axiomatic method and is the earliest example of the format still used in mathematics today, that of definition, axiom, theorem and proof. Euclid arranged them into a single, coherent logical framework (ibid.: 104). The elements were known to all educated people in the West until the middle of the twentieth century and its contents are still taught in geometry classes today (Eves 1990: 141).



Points: In many areas of geometry, such as analytic geometry, differential geometry, and topology, all objects are considered to be built up from points (Gerla 1995).

Line: In analytic geometry, a line in the plane is often defined as the set of points whose coordinates satisfy a given linear equation (Casey 1885). For instance, planes can be studied as a topological surface without reference to distances or angles (Munkres 2000).

Following are five axioms of Euclid:

1. Any two points can be joined by a straight line.
2. Any finite straight line can be extended in a straight line.
3. A circle can be drawn with any centre and any radius.
4. All right angles are equal to each other.
5. If two straight lines in a plane are crossed by another straight line called the transversal, and the interior angles between the two lines and the transversal lying on one side of the transversal add up to less than two right angles, then on that side of the transversal, the two lines extended will intersect (also called the parallel postulate).

Euclid's fifth postulate cannot be proven as a theorem. Euclid himself used only the first four postulates, but was forced to invoke the parallel postulate. In 1823, Janos Bolyai and Nicolai Lobachevski independently realized that entirely self-consistent "non-Euclidean geometries" could be created in which the parallel postulate did not hold.

Archimedes (287–212 BCE) of Syracuse is often considered to be the greatest of the Greek mathematicians; he developed methods very similar to the coordinate systems of analytic geometry. Geometry was connected to the divine for most medieval scholars. The compass in this thirteenth-century manuscript is a symbol of God's act of Creation.

ISLAMIC GOLDEN AGE

The final destruction of the Library of Alexandria at the Muslim conquest of Egypt in 642 CE marks the collapse of classical antiquity in the West, and the beginning of the European "Dark Ages". By the beginning of the ninth century, the "Islamic Golden Age" flourished, the establishment of the "House of Wisdom" in Baghdad marking a separate tradition of science in the medieval

Islamic world, building not only Hellenistic but also on Indian sources. Al-Mahani (820 CE) conceived the idea of reducing geometrical problems such as duplicating the cube to problems in algebra. Thābit ibn Qurra was a Arab mathematician, generalized the Pythagorean theorem, which he extended from special right triangles to all triangles in general, along with a general proof (Sayili 1960: 35-37).

ARABIA

In the Middle Ages, mathematics in medieval Islam contributed to the development of geometry, especially algebraic geometry (Rashed 1994: 35). Three scientists, Ibn al-Haytham, Khayyam and al-Tusi, had made the most considerable contribution to this branch of geometry whose importance came to be completely recognized only in the nineteenth century. The theorem on quadrilaterals, including the Lambert quadrilateral in which three of its angles are right angles, had a considerable influence on the development of non-Euclidean geometry.

CHINA

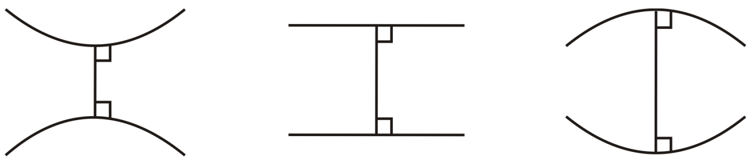
The Chinese knew the relation $3^2 + 4^2 = 5^2$ in the time of Chou Kong (Mikami 1913: 7)[1] (1105 BCE). The first definitive work on geometry in China was the *Mo Jing*, the Mohist canon of the early philosopher Mozi (470–390 BCE). It was compiled after his death by his followers around the year 330 BCE (Needham 1959, vol. 3: 91). However, due to the infamous Burning of the Books in a political manoeuvre by the Qin Dynasty ruler Qin Shihuang (221–210 BCE), multitudes of written literature created before his time was purged. This book included many problems where geometry was applied and included the use of the Pythagorean Theorem. The book provided illustrated proof for the Pythagorean Theorem (ibid.: 22).

EUROPE

The first European attempt to prove the postulate on parallel lines made by Witelo, the Polish scientists of the thirteenth century. The proofs put forward in the fourteeth century by the Jewish scholar

[1] The Kahun Papyrus (2000 BCE) contains four similar relations.

Levi ben Gerson (France). Euclid had stimulated both J. Wallis's and G. Saccheri's studies of the theory of parallel lines. Euclid's fifth postulate, the parallel postulate, is equivalent to Play fair's postulate, which states that, within a two-dimensional plane, for any given line ℓ and a point A, which is not on ℓ, there is exactly one line through A that does not intersect ℓ. In hyperbolic geometry, by contrast, there are infinitely many lines through A not intersecting ℓ, while in elliptic geometry, any line through A intersects ℓ.



INDIA

Geometry arose independently in India, with texts providing rules for geometric constructions appearing as early as the third century BCE, both in Vedic and Jaina cultures.

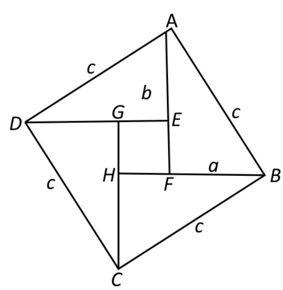
Geometry in Vedic Culture

Indian mathematicians also made many important contributions to geometry. The *Śatapatha Brāhmaṇa* (third century BCE) contains rules for ritual geometric constructions that are similar to the Śulbasūtras. According to Hayashi, the Śulbasūtras contain the earliest extant verbal expression of the Pythagorean Theorem, although it had already been known to the old Babylonians. In the Bakṣālī manuscript, there are a handful of geometric problems. The *Āryabhaṭīya* (499 CE) includes the computation of areas and volumes, he stated his famous theorem on the diagonals of a cyclic quadrilateral and complete description of rational triangles (i.e. triangles with rational sides and rational areas) (Hayashi 1995: 121-22).

The Śulbasūtras in the Vedas is a manual of geometrical constructions (Murthy 1992: 1). The *Taittirīya Saṁhitā* of the *Yajurveda* gives the measurement of *mahāvedī* with a right angle of sides 15, 26 and hypotenuse 39. Kātyāyana gives the construction of a right angle triangle with sides $\frac{n^2-1}{a}\ a$, $n\,a$, and a hypotenuse of length $\frac{n^2+1}{a}\ a$. Such construction was used in building the *vedī*s using the properties of similar triangles. It is surprising to find that an instrument was actually used for drawing circles in the Indus Valley as early as 2500 BCE (Mackay 1938). The date of oldest Śulbasūtras is said to be eight century BCE.

The theorem (Murthy 1993: 155-58) stating that the square on the hypotenuse of a right angle triangle is equal to the sum of the square on its sides has been explicitly stated in the Śulbasūtras. It is attributed to Pythagoras (540 BCE). We call it hypotenuse theorem. Many different proofs have been given. We consider the proof (Amma 1979: 133) given by Bhāsakra. Twice the product of the *bhujā* and *koṭi* combined with the square of their difference will be equal to the square of the side (hypotenuse) (*Bhāsakra Bījagaṇita* 129):

दोः कोट्यन्तरवर्गेण द्विघ्नो घातः समन्वितः ।
वर्गयोगसमः स स्याद् द्वयोरव्यक्तयोर्यथा।।



Draw a square *ABCD* of each side of length *c* units. Draw a perpendicular from point *A*, *B*, *C* and *D* as shown in the diagram which meets at *G*, *H*, *E* and *F*. The length $AF = BH = CG = DE = a$ and the length of $AE = BF = CH = DG = b$. therefore, $GE = EF = FH = HG = a - b$.

The four triangles are all congruent and the area of each triangle = $\frac{1}{2}\, a \cdot b$.

Therefore, the sum of the area of four triangles = $2\,ab$.

Area of the square *ABCD* = c^2. Area of the small square *GHEF* = $(a - b)^2$.

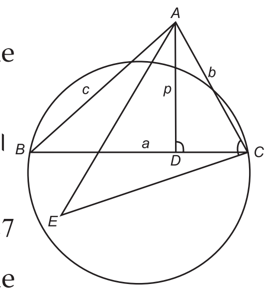
Now area of square *ABCD* = sum of the area of four triangles + area of the small square *GHEF.*

Therefore, $c^2 = 2ab + (a - b)^2$, simplifying we get $c^2 = a^2 + b^2$ which proves the theorem. Proof given by Leonardo da Vinci and Euclid are lengthy. Its proof (Murthy 1993: 158) is also given in the *Yuktibhāṣā*, commentary on the *Tantra Saṁgraha*.

In the *Taittirīya Saṁhitā* (2000 BCE) we find $36^2 + 15^2 = 39^2$.

The method for finding the area of a triangle (Datta 1932: 96) that was known in *Śulba.*

Area of triangle = (base × altitude), by Śrīdhara the area of the triangle $\Delta = \sqrt{s(s-a)s(s-b)s(s-c)}$ where s is semi-perimeter of the triangle.



Derivation of relation $\frac{abc}{4R} = \Delta$ from the Vedic Text (Murthy 1993: 169):

त्रभुजस्य वधो भुजयोद्विगुणित लम्बोध्दृतो हृदयरज्जुः।
सा द्विगुणा त्रिचतुर्भुज कोण स्पृग्वृत विष्कम्भः।।

– ब्रह्मस्फुट सिद्धान्तः XII.27

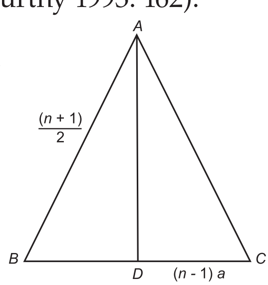
The product of the two sides of a triangle divided by the altitude is equal to the radius of the circle that passes through the three vertices of the triangle, i.e. $\frac{bc}{2P} = R$. Here b and c are two sides of the triangle and p is the altitude. R is the radius of the circle circumscribing the triangle. Draw a triangle ABC of sides a, b and c. Draw AD altitude of length p. Now draw a circle passing through three points ABC. Draw a diameter $2R$ from point A meeting at point E of the circle.

Consider the two triangles ABD and AEC. Angle ABD = angle AEC, and Angle ADB = angle $ACE = \frac{\pi}{2}$. Therefore, triangles ABD and AEC are similar.

Hence

$$\frac{AB}{AE} = \frac{AD}{AC} \text{ or } \frac{c}{2R} = \frac{P}{b} \text{ or } \frac{bc}{2p} = R, \text{or } \frac{abc}{2p} aR \text{ or } \frac{abc}{4p} = \frac{1}{2} aR \text{ or } \frac{abc}{4R} = \frac{1}{2} ap,$$

which is the area of triangle or $\frac{abc}{4R} = \Delta$. Kātyāyana gave the method of construction of a right angle triangle (Murthy 1993: 162).



यावत्प्रमाणनि समचतुर श्राण्येकीकर्तु चिकीर्षेत्
एकोनानि तानि भवन्ति तिर्यक् द्विगुणान्येकत एकाधि
ाकानि। त्र्यस्त्रिर्भवति तस्येषु स्तात्करोति।।

If n squares of side a are to be combined, we have to construct an isosceles triangle ABC with $(n - 1)\ a$ as base and $\frac{n+1}{2}\ a$ as other side. *AD* the altitude is drawn. Then *AD* is the side of the square whose area will be na^2.

To derive the rectangle contained by two sides of a triangle is equal to the rectangle contained by the circum-diameter (Rao 1994: 118) and the altitude to the base, i.e. $AB \cdot AC = AD \cdot AE$.

Interpretation is "as many squares of equal side as you wish to combine into one of the transverses line will be one less than that and twice aside will be one more than that" (Datta 1932: 72-73).

For $BD = ½\, BC = \frac{n+1}{2}\, a$, $AB = AC = \frac{n+1}{2}\, a$, by the Hypotenuse theorem $AD^2 = AB^2 - BD^2 = \left\{\frac{(n+1)}{2}a\right\}^2 - \left\{\frac{(n-1)}{2}a\right\}^2 = na^2$, $AD = \sqrt{na}$.

We can construct a right angle triangle of sides $\frac{(n-1)}{2}a, \sqrt{na}$ and $\frac{(n+1)}{2}a$, or if we put $n = m^2$, we get sides = $\frac{(m^2-1)}{2}a, ma$ and $\frac{(m^2+1)}{2}a$, such geometrical idea was used in the construction of a *vedī*.

Baudhāyana and Āpastamba list several right angle triangles of different measurements (Murthy 1993: 162) (triplets).

Āpastamba	*Baudhāyana*
(15, 36, 39)	(3, 4, 5)
(12, 16, 20)	(5, 12, 13)
(15, 20, 25)	(8, 15, 17)
(5, 12, 13)	(7, 24, 25)
(8, 15, 17)	(12, 35, 37)
(12, 35, 37)	(15, 36, 39)

Early Schools of Geometry

Most notable were the schools of Baudhāyana, Āpastamba and Kātyāyana. The Hindu Geometry (Datta and Singh 1980: 121) originated in a very remote age in need of the construction of the altars. The Hindu geometry did not make much progress in the post-Vedic period (Datta 1929: 479). Al-Biruni, a Persian mathematician and traveller, made an attempt to introduce Euclid's elements in India and later in the Mughal period, it was introduced (Law 1916: 84). Hindu name's for geometry – the earliest name was *śulba*. In the *Mānavaśulba* and *Maitrāyaṇīyaśulba* we get the name *śulba vijñāna* for the science of geometry. Later, Hindu geometry[2] was known as *kṣetragaṇita*. The treatment of plane figures is called *kṣetragaṇita*. There is a general recognition that Indian mathematicians of ancient and medieval time gave only rules and never bothered about their proof which is not completely true. G.R. Kaye (1914: 327) remarks: "The later Indian mathematicians completely ignored the mathematical content of the Śulbasūtras."

Geometry in Jaina Culture

The epithet (nick name) *kṣetragaṇita* occurs in the works of Siddhasena Gaṇi (550). It was also called *rekhāgaṇita* by Jagannātha (1718) and *kṣetramiti* by Bāpūdeva Śastrī. In Jaina works, we find the name *rajju* (Datta 1930: 126). The classification of quadrilaterals is found in the Jaina text *SūryaPrajñapti*. They are *sama caturbhuja* (square), *āyata caturbhuja* (rectangle), *dvisama-caturbhuja* (isosceles trapezium), *trisama caturbhuja* (equitrilateral trapezium) and *viṣama caturbhuja* (quadrilateral of unequal sides). Circle was termed as *maṇḍala*. In the *Sūrya-Prajñapti* eight types of quadrilaterals are given. The geometry of a circle and a straight line is the geometry of the Jambū Island and its symmetric mountains. The Jaina schools carried on an exhaustive campaign to measure every object in various coordinate frames.[3] The *Sūtrakṛtāṇgasūtra* mentions that geometry is the lotus in mathematics and the rest is inferior. In the *Prajñāpanāsūtra* (92 BCE) by Śyāmācārya, references of solid geometry were given by the following *gāthā*:

> जे सण्ठाणपरिणया ते पंजविहा पण्णप्ता तं जहा परिमण्डल सण्ठाणपरिणया,
> तं ससण्ठाणपरिणया, चऊरंससण्ठाण परिणया आयत सण्ठाण परिणया।

Geometrical Results from Jyotiṣakaraṇḍaka Based on Sūrya-Prajñapti

Here a is the length of arc and h is the height between chords, d is the diameter and c is the length of the chord of a circle. The following formulae are mentioned in this text.

---

[2] The *Gaṇitasāra Saṁgraha* of Mahāvīrācārya (850 CE).

[3] *Śrutaskandha*, ch. I, V.154.

Table 23.1: Some Prominent Mathematicians of the Prākṛta Canonical Class and Their Works on Geometry

Mathematician	*Period*	*Major Work*	*Language*
Sudharma Svāmī	300 BCE	*Bhagavatīsūtra*	Ardhamāgadhī
Yativṛṣabha	176–609 CE	*Tiloyapaṇṇattī*	Śaurasenī Prākṛta
Umāsvāti	Fourth century CE	*Jambūdvīpasamāsa, Tattvārthādhigama-sūtra Bhāṣya*	Sanskṛta
Jinabhadragaṇi	600 CE	*Bṛhatkṣetrasamāsa* and *Laghukṣetra-samāsa*	
Akalaṅka	Seventh century CE	*Tattvārthāvṛtikā*	
Vīrasena	816 CE	*Dhavalā*	Śaurasenī Prākṛta
Nemicandra	981 CE	*Trilokasāra, Siddhāntacakravatī*	Śaurasenī Prākṛta
Padmanandi	1000 CE	*Jambūdvīpapaṇṇatti-Saṁgaho*	Śaurasenī Prākṛta

$$c = \sqrt{4h(d-h)},\ a = \sqrt{6h^2 + c^2},\ h = \sqrt{\frac{a^2 - c^2}{6}},\ c = \sqrt{a^2 - 6h^2}.$$

Circumference of circle = $\sqrt{10d^2}$.

Area of circle = $\dfrac{\text{Circumference} \times d}{4}$.

Geometrical Results from Tattvārthādhigmasūtra Bhāṣya

$$h = \frac{1}{2}\left(d - \sqrt{d^2 - c^2}\right)$$

$$d = \frac{\frac{c^2}{4} + h^2}{h}$$

All these results are also given in the *Jambūdvīpasamāsa* by Umāsvāti and in the *Laghu Saṁghāyaṇī* by Haribhradha Sūri.

Geometrical Results from the Jaina School of Mathematics

In *Trilokasāra* V.17, it is mentioned that the circumference of a circle is obtained by multiplications of diameter with three and area is equal to one-fourth of diameter with circumference. If we take approximation of the square root and apply to the result of the area of circle we get or

$$A \approx \left(d^2 - \frac{d^2}{4}\right) + \left(d^2 - \frac{d^2}{4}\right) \times \frac{1}{18}$$

By simple manipulation.

This type of modification was also available outside India (first century BCE). The area of circle was calculated by Heron of Alexandria (Waerden 1983: 18) using

$$A \approx 3\left(\frac{d^2}{2}\right) + \frac{1}{7}\left(\frac{d^2}{2}\right).$$

In 150 CE, Nehemiah, a Hebrew Rabbi (Beckmana 1974: 76) gave the formula for area

$$A = d^2 - \frac{d^2}{7} - \frac{d^2}{14}.$$

Geometrical Results from the Bhāṣya of the Tattvārthādhigamasūtra

Let us denote the area of a circle as A, d its diameter, r its radius, s as arc of its segment whose height is h, c the chord and p the circumference. The formulae are:

1. $p = \sqrt{10d^2}$.
2. $c = \sqrt{4h(d-h)}$.
3. $s = \left(\sqrt{6h^2 + c^2}\right)$.
4. $h = \frac{1}{2}\left(d - \sqrt{d^2 - c^2}\right)$.
5. $d = \left(h^2 + \frac{c^2}{4}\right) \div h$.
6. $A = \frac{1}{2}p.d$.

The part of the circumference of the circle between two parallel chords is half the difference between the corresponding arcs. All the above relations are also available in the *Jambūdvīpasamāsa* with the exception of (4). It instead is $h = \sqrt{\frac{(s^2 - c^2)}{6}}$.

In the *Gaṇitasāra Saṁgraha* it is given, s (gross) = $\sqrt{(5h^2+c^2)}$, s (fine) = $\sqrt{(6h^2+c^2)}$.

In Greek Heron of Alexandria (c + 200), we find (Heath 1921, vol. 2: 331) $s=\sqrt{(4h^2+c^2)}+\frac{1}{4}h$.

The Chinese, Chien Huo (1075 CE) gives (Mikami 1913: 62) $s = \sqrt{(4h^2+c^2)}+\frac{1}{4}h$. Similar formulae occur in the *Kṣetrasamāsa* and the *Laghukṣetra-samāsa*.

Geometrical Results from Tiloyapaṇṇattī

In the words of T.A. Saraswati Amma (1979: 76): "First four *mahādhikāra*s of *Tiloyapaṇṇattī* is a storehouse of mathematical formulae". The author had given formulae for finding the area of different geometrical figures, circumference of circle, length of the chord; the following formulae are available in the *Tiloyapaṇṇattī*.

P – circumference, c – chord, h – height of the chord from centre, s – arc, A – area, d – diameter and r – radius

1. $P = \sqrt{10d^2}$. [(v. 4.6],
2. (Chord of a quadrant arc)^2 = $2r^2$ [v. 4.70]
3. $C = \sqrt{4\left[\left(\frac{d}{2}\right)^2-\left(\frac{d}{2}-h\right)^2\right]}$ [v. 4.180] J.P. gives the rule
4. $c = \sqrt{4.h(d-h)}$ [v. 2.23; 6.9]
5. s = $\sqrt{2\left[(d+h)^2-(d)^2\right]}$ [v. 4.181], J.P. gives the rule
6. $s = \sqrt{6(h^2)+(c^2)}$ [v. 2.24, 29, 6.10]
7. $h=\frac{d}{2}\left[\frac{d^2}{4}-\frac{c^2}{4}\right]^{\frac{1}{2}}$ [v. 4.182]. Here J.P. means *Jambūdvīpa Prajñapti*.

The *Trilokasāra* furnishes the following formulae (Kapadia 1937: XLIV).

1. p (gross) = 3 d and p (subtle) = 10d (v. 311)
2. A = 1/3 pd (v. 311)
3. r = 9/16 (side of square of equal area) or = π (16/9)^2 (v. 18)
4. c^2 = 4h (d – h)
5. s^2 = 6h^2 + c^2 (v. 760)

6. $d = \dfrac{c^2 + 4h^2}{4h}$ (v. 761)
7. A (gross) $= \sqrt{10}c\dfrac{h}{4}$ (v. 762)
8. $c^2 = S^2 - 6h^2$ (v. 766)
9. $s^2 = 4h\left(d + \dfrac{h}{2}\right)$
10. $h = \sqrt{(s^2 - c^2 \div 6)}$ (v. 763)
11. $d = \dfrac{1}{2}\left(\dfrac{a^2}{2h} - h\right)$ (v. 765)
12. $d = \dfrac{1}{2}\left(d - \sqrt{d^2 - c^2}\right)$ (v. 764)
13. $h = \sqrt{d^2 + \dfrac{1}{2}s^2 - d}$ (v. 765)

Similarly, the *Gommasāra* contains the formulae about volumes of a prism, as base into height. The volume of a sphere is equal to 9/2 (radius). There is a *gāthā* 1.24 of the *Jambūdvīpasamāsa* to find the area of the circle:

विक्खंभचदुब्भागेण संगुणं होई परिधि परिमाणं ।
पद्रगदं खेत्तफलं लद्धं रविमंडलाण तहा ।।

In the above *gāthā* the formula for the area of a circular thing is given.

The area is $A = c \times \dfrac{d}{4}$, where c is the circumference and d is the diameter. We also find the formulae for the chord, length of the arc, height of chord from the lowest point of the circle and other result. These types of results are also given in the *Lokavibhāga* text.

चदुगुणइसूहि भजिदं जीवावग्गं पुणों वि इसुसहिदं ।
परिमंडलखेत्तस्स दु विक्खंभं होइ णायव्वं ।।

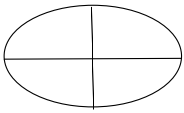
– *Gāthā* 2.26

Explanation: In the above *gāthā*, the formula for the diameter of the circle is given as:

$$\text{dia} = \frac{(\text{chord})^2}{4(\text{height})} + \text{height}.$$

In the *Sūrya-Prajñapti* it was known by *Viṣamacakravāla*. Menaechmus (*c*.350 BCE) (Heath 1921, vol. I: 11) obtained ellipse by cutting an acute angled cone by plane perpendicular to it and hyperbola from right and obtuse-angled cone.

Perimeter of the ellipse = $\sqrt{4a^2 + 6b^2}$ = P (Parameter)



Ellipse

And area of the ellipse = $P \times \frac{2b}{4} \frac{b}{2} \sqrt{4a^2 + 6b^2}$.

Takao Hayashi (1990: 5) points that the terminology for the breadth ($2b$) and for the length ($2a$) implies the condition $b \leq a$.

Geometry in Bhagavatīsūtra

The geometrical figures such as triangle, quadrilateral, circle, rectangle and ellipse are mentioned in it.[4] This text has 656 *gāthās* containing mathematical results of solid geometry and plane geometry. Malayagiri wrote commentary on it (Upadhyaya 1971: 241). Jinabhadragaṇi Kṣamāśramaṇa (609 CE) wrote this mathematical book. Jinabhadragaṇi explains by a mathematical formula in *gāthā* 122 how to find the area of different regions of Jambūdvīpa. The author gives a method to find circular area between two parallel chords of the circle in the *gāthā* (Gupta 1987: 60-62). Given the length of small chord $AB = a$ and length of big chord $CD = b$, the distance between the two chords, $LN = h$.

According to the mathematical law given, the area of circular region $ABFDCEA$ is:

$$K = \left[\sqrt{\frac{1}{2}(a^2 + b^2)} \right] \times h$$

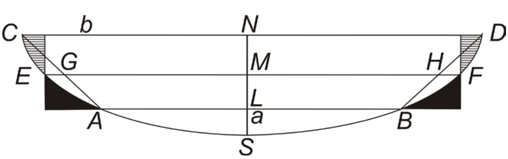


fig. 23.1: Area of Circular Region

The detail of this result is as follow. The area of trapezium $ABHDCGA$ inside the required circular part[5] is $T = 1/2\ (a + b)\ h$. However, this is less than the required area. So using

[4] *Bhagavatīsūtra*, *śataka* 24, *uddeśaka* 3.

[5] *Bṛhatkṣetrasamāsa*, A-1, *gāthā* 64.

$$\left[\frac{a+b}{2}\right]^2 = \frac{1}{2}\left[a^2+b^2\right] - \left[\frac{a-b}{2}\right]^2,$$

$$(GH)^2 = \frac{1}{2}\left[a^2+b^2\right] - \left[\frac{a-b}{2}\right]^2.$$

However, the first result gives more value of the area than its exact value. The explanation is as follows – Let the length of the chord = C and the Height of the segment = g. Then using this result of ancient time we have $4g\ (2R - g) = C^2$, where R is the radius of the circle. This formula was known to Jinabhadragaṇi[6] using this result; we can find the length of the chord EF that is the middle chord between AB and CD chords. Its length is – $(EF)^2 = 1/2(a^2 + b^2) + h^2$. Therefore, we can consider the effective average length of the chord, which is approximately taken as $\sqrt{\frac{1}{2}(a^2+b^2)}$.

When it is multiplied by h we get the required results which is same as given by Jinabhadragaṇi.

Table 23.2: Comparison of Circumference and Area of Circle in Different Jaina Texts

Text *Formula*	*TP*	*GSS*	*JPS*	*TS*	*Modern Value*
Circum-ference of the circle	$c = \sqrt{d^2 \times 10^2}$	$C = 3 \times d$ $c = \sqrt{d^2 \times 10^2}$	$C = \sqrt{d \times d \times 10}$	$C = 3 \times d$ $C = \sqrt{10} \times d$	$C = 2\pi r$
Area of the circle	$A = C \times \frac{d}{4}$	$A = 3 \times \left(\frac{d}{2}\right)^2$	$A = \sqrt{10\left(\left(\frac{d}{2}\right)^2\right)^2}$ $A = C \times \frac{d}{4}$	$A = C \times \frac{d}{4}$	$C = 2\pi r$

6 *Bṛhatkṣetrasamāsa*, A-1, *gāthā* 36.

Table 23.3: Comparisons of the Relation between Chord, Height of the Chord, Arc and Diameter of a Circle in Different Jaina Texts

Text *Formula*	*TP*	*JPS*	*TS*	*LV*
Chord in terms of *h* and *d*	$c=\sqrt{4\left[\left(\frac{d}{2}\right)^2-\left(\frac{d}{2}-h\right)^2\right]}$	$c=\sqrt{(d-h)\times h\times 4}$	$c^2=4\times h(d-h)$	$c=\sqrt{(d-h)\times 4\times h}$
h in terms of *c* and *d*	$h=\frac{d}{2}-\sqrt{\left(\frac{d}{2}\right)^2-\frac{1}{4}c^2}$	$h=\frac{d-\sqrt{d^2-c^2}}{2}$	$h=\frac{d-\sqrt{d^2-c^2}}{2}$	–
d in terms of *c* and *h*	$d=\frac{c^2}{4.h}+h$	$d=h+\frac{c^2}{4\times h}$	$d=\frac{c^2+4\times h^2}{4\times h}$	–
a in terms of *c* and *h*	$a^2 = 6h^2 + c^2$	$a^2 = 6 \times h^2 + c^2$	$a^2 = 6 \times h^2 + c^2$	$a^2 = 6 \times h^2 + c^2$

Here *C* – circumference, *d* – diameter and *A* – area

Length of Chord = *c*, Diameter = *d*, Length of the arc = *a*

Height of the chord from the lowest point of the circle. = *h*,

TP = Tiloyapaṇṇattī, JPS = Jambūdvīpapaṇṇattī Saṁgaho, TS = Trilokasāra, LV = Lokavibhāga, GSS – Gaṇitasāra Saṁgraha

Modern Geometry

In the early seventeenth century, there were two important developments in geometry. The first and most important was the creation of analytic geometry by René Descartes (1596–1650) and Pierre de Fermat (1601-65). Since then, and into modern times, geometry has expanded into non-Euclidean geometry and manifolds, describing spaces that lie beyond the normal range of human experience. This was a necessary pioneer to the development of calculus. The second geometric development of this period was the systematic study of projective geometry by Girard Desargues (1591–1661). Projective geometry is the study of geometry without measurement, just the study of how points align with each other (Rosenfeld and Yausehkeviten 1996, vol. 2: 470).

The Eighteenth and Nineteenth Centuries: Non-Euclidean Geometry

The very old problem of proving Euclid's Fifth Postulate, the "Parallel Postulate", from his first four postulates had never been forgotten. Giovani Girdamo Saccheri (1701), John Heinrich Lambert (1760), and Adrien Marie Legendre (1799) each did excellent work on the problem in the eighteenth century. Beginning to suspect that it was impossible to prove the Parallel Postulate, they set out to develop a self-consistent geometry in which that postulate was false. In this they were successful, thus creating the first non-Euclidean geometry.

In the twentieth century, David Hilbert (1862–1943) employed axiomatic reasoning in an attempt to provide a modern foundation of geometry. Analytic geometry applies methods of algebra to geometric questions, typically by relating geometric curves to algebraic equations. Euler called this new branch of geometry *geometria situs* (geometry of place), but it is now known as topology. Topology grew out of geometry, but turned into a large independent discipline.

Application

Geometry has applications in many areas, including cryptography, the art of writing or solving codes and in string theory (string theory is a theoretical framework in which the point-like particles of particle physics are replaced by one-dimensional objects called strings). Discrete geometry is concerned mainly with questions of relative position of simple geometric objects, such as points, lines and circles. Euclidean geometry also has applications in computer science, crystallography (crystallography is a technique used for determining the atomic and molecular structure of a crystal) and various branches of modern mathematics. An important area of application is number theory. In ancient Greece the Pythagoreans considered the role of numbers in geometry. Since the nineteenth century, geometry has been used for solving problems in number theory, for example, through the geometry of numbers or, more recently, scheme theory, which is used in Wiles's proof of Fermat's Last Theorem.

Conclusion

We found that geometry is well explained in all the philosophical and mathematical texts of different cultures. Mathematicians and others developed geometry for different purposes. There was a pervasive fascination with geometrical results. It will motivate further studies and research of ancient and medieval geometry. We have shown that geometry grew independently in different cultures. Indians had also good knowledge of geometrical calculations and their approach was scientific. We should not forget that all these accomplishments were made in the absence of the modern mathematical techniques. Indian ancient texts remained unexposed to the Western countries due to several reasons and the history was written by English so no importance was given to Indian mathematicians by foreigners (Dange 1972). Certainly, it seems that Indian contributions to geometry has not been given due acknowledgement until very recently in modern history of mathematics.

References

Amma, T.A. Sarasvati, 1979, *Geometry in Ancient and Medieval India*, Delhi: Motilal Banarsidass.

Beckmann, Petr, 1974, *A History* of π, New York: St. Martine's Press.

Benjanin, Boyer Carl, 1968, "Ionia and Pythagoreans", *A History of Mathematics*, USA: John Wiley and Sons.

Boyer, Carl Benjanin, 1991, "Ionia and the Pythagoreans", A History of Mathematics, New York: John Wiley.

Bṛhatkṣetrasamāsa, Acharya Jinbhadragani-krita, Malayagiri krit Vritti Sahit, Bhavanagar, 1977.

Casey, John, 1885, *Analytic Geometry of the Point, Line, Circle, and Conic Sections,* London: Longmins Greens Co.

Datta B.B., 1929, "Scope and Development of the Hindu Gaṇita", *Ind. His. Quart.*, V.

———, 1930, "Origin and History of the Hindu Names for Geometry", *Qullen and Studien sur Gesehishte der Mathematik*, B.I.: 113-19.

———, 1932, *The Science of Śulba*, Calcutta: Calcutta University Press.

Datta B.B. and A.N. Singh, 1980, "Hindu Geometry", *IJHS*, 15(2).

Dange, S.A., 1972, "India 5th edn", New Delhi: People's Publishing House.

Eves, Howard, 1990, *An Introduction to the History of Mathematics*, Philadephia, PA: Saunders.

Forbes, Peter, 2012, "All That Useless Beauty, The Guardian", *Review: Non-fiction*, 11 February.

Friberg, J., 1981, "Methods and Traditions of Babylonian Mathematics: Plimpton 322, Pythagorean Triples, and the Babylonian Triangle Parameter Equations", *Historia Mathematica*, 8: 277-318.

Gerla, G., 1995, "Pointless Geometries", in *Handbook of Incidence Geometry: Buildings and Foundations*, ed. F. Buekenhout and W. Kantor, pp. 1015-31, Amsterdam: North Holland Publishing Co..

Gupta, Radhacharan, *Jinabhadragaṇi ke eka Jyāmitīya Sūtra kā Rahasya*: *Astha and Chintan*, Acharya Desh Bhushan Abhinandan Granth, Delhi: Jain Prachya Vidya.

Hayashi, Takao, 1990, "Narayana's Rule for a Segment of Circle", *Ganita Bharati*, 12: 1-9.

Hayashi, Takao, 1995, *The Bakhshali Manuscripts: An Ancient Indian Mathematical Treatise*, Groningen Oriental Studies, Jan.

Heath, Thomas, 1921, *A History of Greek Mathematics*, vols I-II, London: Oxford University Press, repr 1960.

Hogben, Lancelot, 1967, *Mathematics for Millions*, London: Pan Macmillan.

Kapadia, H.R., 1937, *Bhagavatīsūtra*, śataka 24 , uddeśaka 3, Introduction of *Gaṇita Tilaka*, p. XLIV, Baroda: Oriental Research Institute.

Kaye, G.R., 1914, *Indian Mathematics*, Calcutta Shimla: Thacker, Spink and Co.

Lambert, Johann Heinrich, 1760, *Photometria Book*.

Law, N. N., 1916, *Promotion of Learning in India during Muhammadan Rule*, London: Longmans Green & Co..

Legendre, Adrien Marie, 1799, *Elements of Geometry*.

Mackay, E.J.H., 1938, *Further Excavation at Mohenjo-daro,* Delhi: The Manager Controller of Publications Government of India.

Mikami, Yoshio, 1913, *The Development of Mathematics in China and Japan*, Leipzig, B.G. Teubner and Williams and Norgate, London.

Murthy, T.S. Bhanu, 1993, *A Modern Introduction to Ancient Indian Mathematics*, New Delhi: New Age International Publishers.

Munkres, James R., 2000, *Topology*, vol. 2, Upper Saddle River, NJ: Prentice Hall.

Needham, Joseph, 1959, *Science and Civilization in China*, vol. 3: *Mathematics and the Sciences of the Heavens and the Earth*, Taipei: Caves Books.

Neugebauer, Otto (ed.), 1969, *The Exact Sciences in Antiquity 2*, chap. IV: "Egyptian Mathematics and Astronomy", pp. 71-96, New York: Dover Publications.

Rao, S. Balachandra, 1994, *Indian Mathematics and Astronomy*, Bongalore: Jnana Deepa Publications.

Rashed, R., 1994, *The Development of Arabic Mathematics: Between Arithmetic and Algebra*, London: Kluwer.

Risi, Vincenz De, 1701, *Giovani Girdamo Saccheri Euclid*, *Vindicated*, tr. G.B. Halasted.

Rosenfeld, Boris A. and Adolf P. Youschkevitch, 1996, "Geometry", in *Encyclopedia of the History of Arabic Science*, ed. Roshdi Rashed, vol. 2, pp. 447-94, London and New York: Routledge.

Saccheri, Giovani Girdano, 1701, *Euclid*, Vindicated Vincenz de Disi, tr. G.B. Halasted.

Sayili, Aydin, 1960,"Thabit ibn Qurra's Generalization of the Pythagorean Theorem", *Isis*, 51(1): 35-37.

Schmidt, W., R. Houang and L. Cogan, 2002, "A Coherent Curriculum", *American Educator*, 26(2): 1-18.

Steen, L.A., 1998, "The Science of Patterns", *Science*, 240: 611-16, Summary at ascd.org.

Stewart, Ian, 2001, "What Shape is a Snowflake", *in Magical Number in Nature*, pp. 48-49, New York: Henry Holt and Company.

Tattvārthādhiganasūtra, Acharya Umaswati, English tr. K.P. Modi 1st Part, 1903 and H.L. Kapadia, 2nd part, 1926, Baroda: Gaikwad Oriental Institute Series.

Turner, Martin J., Jonathan M. Blackledge and Patrick R. Andrews, 1998, *Fractal Geometry in Digital Imaging*, Cambridge, MA: Academic Press.

Trilokasara, Acharya Nemichandra Siddhant Chakravarti, 2nd edn, Gannor, Haryana: Sahitya Bharati Prakashan, 2005.

Upadhyaya, B.L. 1971, *Prācīna Bhāratīya Gaṇita*, Delhi: Vigyan Bharati.

Waerden, van der, 1983, *Geometry and Algebra in Ancient Civilizations*, New York: Springer-Verlag.

28

Life and Works of T.A. Saraswati Amma and Suggestions for Future Work in Geometry

P.S. Chandrasekaran

Abstract: T.A. Saraswati Amma's early life, education and academic career have been briefly described. Her modern approach to prove some of the *sūtra*s and her systematic chronicling of the developments in geometry in India from ancient times to early seventeenth century have been highlighted. A few suggestions for further work including examples, thereof, have been provided.

T.A. Saraswati Amma, the Sanskrit scholar and mathematician par excellence, who has contributed immensely to the recording of Indian geometry was born as the second daughter of Achyuta Menon and Kuttimalu at Cherpulassery in Kerala in the year 1918.

She had her basic graduation from the University of Madras, with Physics and Mathematics as her main subjects. She then took her MA in Sanskrit from the Banaras Hindu University and MA in English Literature from Bihar.

After her studies, Saraswati Amma worked for a number of years in the Sree Kerala Varma College, Trissur and the Maharaja College, Ernakulam. In the year 1957, she joined the Sanskrit Department of University of Madras as a Government of India

Research Scholar and came under the guidance and mentorship of the great Indologist V. Raghavan.

Raghavan, who clearly saw her huge potential, advised her to take up research in the field of Indian contributions to mathematics.

Saraswati Amma's talent bloomed under Raghavan's watchful eyes and she could bring to bear her considerable erudition in both Sanskrit and mathematics on the texts she laid her hands on. Being a Malayalee helped her considerably as many of the exciting developments in the post-Bhāskara II phase were concentrated in Kerala and she could easily understand and analyse the texts, which were in Malayalam.

Her research work was completed in 1963 and she was awarded a doctorate degree in 1964. Much as she tried, she could not publish her research work till 1979. The book which was published by Motilal Banarsidass under the title *Geometry in Ancient and Medieval India*, drew rare reviews and catapulted her to instant celebrity status. The book traces the History of Indian mathematics from the Vedic times to the early seventeenth century. Besides providing proofs of many mathematical formulae, she also drove home the point that some of the discoveries in India preceded those of the West by three to four centuries.

After retirement from the principal's post at Dhanbad, where she worked last, Saraswati Amma moved to her home in Ernakulam to attend to family work and her aged mother. She shifted to Ottappalam subsequently. Because of family issues she could not continue her research work and breathed her last on 15 August 2000.

It is noteworthy that no subsequent work on Indian geometry has come about, though it is almost forty-two years since her book was first published. This in itself is an ample testimony to the comprehensive and thorough nature of her treatment of the subject.

Saraswati Amma justifiably introduces her book as the third in a series of books on Indian mathematics, succeeding Parts I and

II of the *History of Hindu Mathematics* by Bibhutbhushan Datta and Awadesh Narayan Singh.

The text *Geometry in Ancient and Medieval India* contains ten chapters. In the first chapter which forms the introduction, the author gives a brief history of Indian mathematics beginning with the Vedic period, up to the seventeenth century CE. The author explains that the absence of proof in many of the *sūtra*s is due to the fact that mathematical knowledge for its own sake did not interest the Indian scholars and that the mathematical knowledge was deeply rooted in its applied nature. However, proofs were given in later-day commentaries for many *sūtra*s.

Chapter II is devoted to the "Śulbasūtras". Many important features of the same such as the theorem of the square of the diagonal, construction of squares, rectangles and trapezia, combination and subtraction of areas, properties of similar figures and areas, etc. are explained in great detail.

Chapter III deals with geometry as found in early Jaina canonical texts. The value of $\sqrt{10}$ for π, solid figures, relations between chord lengths, height of chords and areas of segments have been explained in this chapter.

The balance chapters are arranged subject-wise. Chapters IV-VII deal with trapezia, quadrilaterals, triangles and circles.

The chapter on trapezia deals with the treatment of the subject in early Jaina literature as well as by Āryabhaṭa I, Brahmagupta, Mahāvīra and later authors like Śrīpati and Bhāskara.

The chapter on quadrilateral gives a detailed exposition of the cyclic and non-cyclic quadrilaterals and an in-depth discussion on Brahmagupta's treatment as well as analysis by the *Kriyākramakarī Yuktibhāṣā,* etc.

Chapter VIII is on volumes and surfaces of pyramid, formation of a cone, sphere, etc. are dealt with. The surface area and volumes of spheres are derived by integration methods.

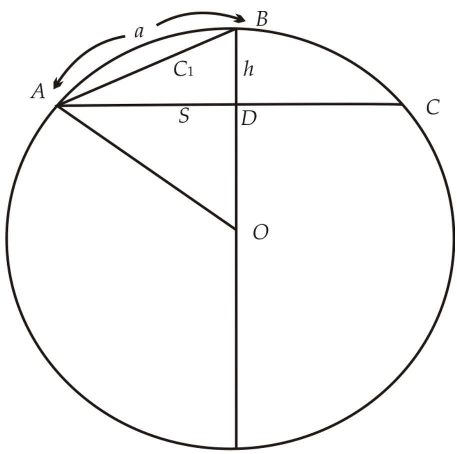
Chapter IX deals with geometric algebra, where the practice of representing and solving algebraic and arithmetical problems geometrically is explained in detail.

Chapter X deals with shadow measurements and calculations which form an important part of astronomy and therefore of mathematics from very early texts.

Saraswati Amma's Methods of Handling Some Important Topics

SEGMENTS OF CIRCLES

One of the important relations in a segment of circle is that connecting the arc length to the height of twice the arc h and the sine chord S.



The expression is

$$a_1 = \sqrt{(h^2 (1 + 1/3) + s^2)}$$

a_1 = arc length AB

h = height of twice the arc, i.e. ABC

s = sine chord.

Saraswati Amma (2007: 180-82) derives the formula by dividing the arc successively into half of the original size till the arc becomes so small, to be considered equal to the chord. The height in each case is expressed in terms of the original height.

A geometric progression is formed and the sum up to ∞ leads to the simple expression for a_1 in terms of s and h.

Two principles of calculus are used here:

i. A very small length of a curve is equal to the chord joining the two end points.
ii. Integration as a sum.

CYCLIC QUADRILATERALS

In his review of the book, the Japanese scholar Michio Yano states that "Saraswati's discussion of the cyclic quadrilaterals treated by Brhamagupta reveals her remarkable competence in dealing with mathematical Sanskrit texts".

The scholar further states that "the proofs of the well-known 'Brhamagupta's theorem' and his formula for the area of the cyclic quadrilaterals are reproduced by the author according to the sixteenth-century works such as the *Tantra Saṁgraha*, *Yuktibhāṣā* and *Kriyākramakarī*.

While deriving the various formulae for a quadrilateral, Saraswati Amma freely uses the facts that:

i. Angle in a semi-circle is a right angle.
ii. Angles in the same segment of a circle are equal and she also feels that perhaps these results were known in India much earlier.

She further derives some trigonometric results, based on *Yuktibhāṣā*, such as:

i. $\sin^2 A - \sin^2 B = \sin(A + B)\sin(A - B)$
ii. $\sin A \sin B = \sin^2(A + B)/2 - \sin^2(A - B)/2$
iii. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

TRIGNOMETRIC AND INVERSE TRIGNOMETRIC SERIES

Yano describes the chapter VII the most remarkable chapter of the geometry in ancient and medieval India "which shows an outstanding aspect of Indian mathematics – the discovery of the infinite series of π, and of sine and cosine series.

Through a lengthy procedure, Saraswati Amma derives the series formulae for sin ø and cos ø, viz.

$\sin ø = ø - ø^3/3! + ø^5/5! \ldots\ldots.$

$\cos ø = 1 - ø^2/2! + ø^4/4!$ ....

The series were known in Europe by the seventeenth century whereas in India they were known as early as in the fourteenth century.

Similarly Saraswati Amma describes a method to evaluate π as a series:

$\pi/4 = 1 - 1/3 - + 1/5 - 1/7$ ...

which was enunciated by Gregory three centuries later in the form of series for ø in terms of tan ø, viz.

$ø = \tan ø - (\tan^3 ø)/3 + (\tan^5 ø)/5$ ...

which can be reduced to the series shown above putting $\tan ø = x$ and $x = 1$.

Major Achievements of Saraswati Amma

1. Saraswati Amma deals with the development of geometry in India right from the period of Śulbasūtras up to early seventeenth century.

   In this respect her book is rightfully a successor to the two volumes by Datta and Singh.

   Though there appear to be no other books published by her, yet this single work places her on a unique pedestal among the scholars of Indian mathematics.

2. Being a Keralite, Saraswati Amma was in an advantageous position to analyse the various Malayalam manuscripts of the post-Bhāskara II phase. She diligently culled out, analysed and compared various approaches in geometry in Indian mathematics including famous works and commentaries.

3. She has used the concepts of algebra and calculus, etc. to illustrate the correctness of some of the formulae from our old texts.

   The surface area and volume of the sphere have been derived using the principles of integration by the author.

4. Though her doctoral thesis was made in 1963, Saraswati

Amma could not get it published in book form till 1979. It shows the strong will and perseverance of Saraswati Amma that she finally succeeded, even without the official funds materializing for publishing the book.

5. In the rarely touched upon field of Indian geometry, Saraswati Amma succeeded, and succeeded remarkably well.

It is a work of such greatness that even after forty-two years of her publication, there has been no sequel to her work.

Scope for Further Research

While it is true that Saraswati Amma has comprehensively dealt with all the features of various geometric figures in her book, there is also scope for further work in areas that the author has only briefly touched upon, due to paucity of time and space. Two such cases are presented here.

One example is the subject of regular polygons inscribed in a circle, where the author, referring to Bhāskara, states that his method of calculating the values of the sides are not known. She also remarks (2007: 192-93) that Gaṅgeśa's method of dividing the circumference into as many equal parts and evaluating the chord corresponding to one division using the sine table does not yield results exactly tallying with those of Bhāskara.

The topic has been dealt with in subsequent literature, viz. Bhāskarācārya's *Līlāvatī* by A.B. Padmanabha Rao (2014: 129-32). Geometric methods have been provided by the *Buddhivilāsini* but only for sides of 3, 4, 6 and 8. Rao has suggested a geometric method for a pentagon while quoting the *Buddhivilāsinī* that the heptagon and nanogon cannot be treated by any geometric procedure.

An attempt is made here to derive the sides of a regular polygon of *n* sides, inscribed in a circle, through simplification and restatement of Bhāskara's *sūtra*s for the chord of a circle. There is a very good closeness of the results obtained to the values stated by Bhāskara.

In *śloka* 219 of the *Līlāvatī*, Bhāskara enunciates the formula for the chord of a circle, thus:

चापोननिघ्नपरिधिः प्रथमाह्वयः स्यात्पञ्चाहतः परिधिवर्गचतुर्थभागः।
आद्योनितेन खलु तेन भजेच्चतुर्घ्न व्यासहतम् प्रथमप्राप्तमिह ज्यका स्यात्।।

The circumference diminished and multiplied by the arc shall be called the *prathamā*. One quarter of the circumference squared multiplied by 5 is to be diminished by the *prathamā*. The *prathamā* multiplied by 4 and the diameter should be divided by the above result. The quotient will be the chord.

Thus, if c is the chord of the arc a and if d and p are diameter and circumference of the circle whose part the arc is

$$c = \frac{4da(p-a)}{5p^2/4-(p-a)a}. \qquad (1)$$

When a regular polygon of n sides is inscribed in a circle, it divides the circle into n equal arcs, each of length p/n.

Substituting this value of a in formula (1):

$$c = \frac{4d(p/n)(p-p/n)}{5p^2/4-(p-p/n)p/n}$$

$$c = \frac{(4dp/n)p(1-1/n)}{5p^2/4-p/n\times p(1-1/n)}$$

$$c = \frac{4dp^2(n-1)/n^2}{5/4p^2-p^2/n^2\times(n-1)}$$

$$c=\frac{4d(n-1)}{5/4n^2-(n-1)}$$

$$c=\frac{16d(n-1)}{5n^2-4(n-1)}. \qquad (2)$$

This formula which does not involve the arc length a and perimeter p can be used to compute the side of the polygon, viz. c.

In *śloka*s 206-08, the *Līlāvatī* lists the lengths and sides of regular polygons of sides 3 to 9 inscribed in a circle of diameter 120,000 units thus:

No. of Sides	*Side Length When d = 1,20,000*
3	103,923
4	84,853
5	70,534
6	60,000
7	52,055
8	45,922
9	41,031

The table below shows the values of the sides as stated by Bhāskara and the values derived by using restated formula, and also the percentage deviations. It may be seen that the derived values in column 3 very well with those of column 2.

No. of Sides	*Length as per Sūtra*	*Length as per Restated Formula*	*Percentage Deviation*
3	103,923	103,788	– 0.130
4	84,853	84,708	– 0.171
5	70,534	70,452	– 0.116
6	60,000	60,000	0.000
7	52,055	52,128	0.140
8	45,922	46,032	0.240
9	41,031	41,172	0.344

The restated formula is thus useful for any n-sided polygon and not limited to 9. The restated formula actually represents $d \sin \pi/n$ and is used as an alternative to looking up the sin tables or evaluation of the side length.

Another case involves a number of series for π attributed to Mādhava where Saraswati Amma states that the series can be got by regrouping the terms of the series

$\pi/4 = 1 - 1/3 + 1/5$ ..., etc.

but does not indicate how the regrouping is to be done.

We can use a generalized method, using a common approach for writing the n^{th} term, splitting it and then writing down the sum of the series.

Conclusion

Raghavan in his introduction to Saraswati Amma's book says that the material available should be interpreted in terms of modern knowledge in the concerned sciences. It is in this respect that Saraswati Amma's contribution should be assessed, as she was one of a kind combining in herself deep knowledge of Sanskrit, Malayalam and English and an equal command over mathematics and sciences. Her book thus marks a milestone in the understanding and appreciation of Indian mathematics.

Further, Saraswati Amma has stated in her work that irregular shapes in geometry have not been taken up in her book. These may be attempted. Some of her proofs may also be derived from the use of trigonometric formulae wherever possible.

References

Amma, T.A. Saraswati, 2007, *Geometry in Ancient and Medieval India*, Delhi: Motilal Banarsidass.

Rao, A.B. Padmanabha, 2014, *Bhāskarācārya's Līlāvatī*, Ernakulam: Chinmaya International Foundation Shodha Sansthan.

Rao, S. Balachandra, 2017, *Indian Mathematics and Astronomy*, Bengaluru: Bhavan's Gandhi Centre of Science & Human Values, Bharatiya Vidya Bhavan.

29

Indian Math Story
Website dedicated to History of Indian Mathematics
https://indianmathstory.com

Pattisapu Sarada Devi

Abstract: I shall present here a website: https://indianmathstory.com, developed by me in 2018, which is a chronicle of my efforts in history of Indian mathematics. I am positive that this will encourage and motivate the students and researchers of this subject. This website consists of:

a. Various conferences on this subject that I participated from year 2000 onwards.

b. Titles of several reference books and names of the journals.

c. Names of several resource persons – questions/puzzles on history of Indian mathematics.

d. Links of Resource Videos – Mathematical Tourism in India.

e. About the play *Journey through Maths*: *The Crest of the Peacock* that I have developed and staged.

f. Honours programmes that I had conducted at St. Xavier's College, Mumbai.

This website is a continuous saga as it has room for various additions in the future. I welcome all your valuable suggestions.

Introduction

In 2000, World Mathematics Year, I had developed a play on history of Indian mathematics. I thank the staff of HBCSE, Mumbai and School of Mathematics, TIFR, Mumbai for their support and encouragement for developing the play, especially for the research material. A token amount was obtained from HBCSE, Mumbai for the stationary and reference material for the research. The script of the play, *Journey through Math*: *The Crest of the Peacock,* developed with the help of the students of St. Xavier's College, Mumbai. It was staged at several places later on. One of the observations was that there is zero amount of awareness of history of Indian mathematics not only among the students but also among mathematics teachers. It is like roots of the present generation had almost been cut from their heritage. Hence, my interest in this unattended subject grew and that made me start attending the various conferences held on this subject, which is the history of Indian mathematics, from 2000 onwards. I have also conducted honours programme for three years at St. Xavier's College where I used to work as a lecturer for more than twenty years. After my retirement, I found the necessity to document all my efforts in this subject. Hence this website: https://indianmathstory.com.

About the Website

This website contains altogether thirty-nine tabs including fourteen main tabs and various sub-tabs. This number is dynamic.

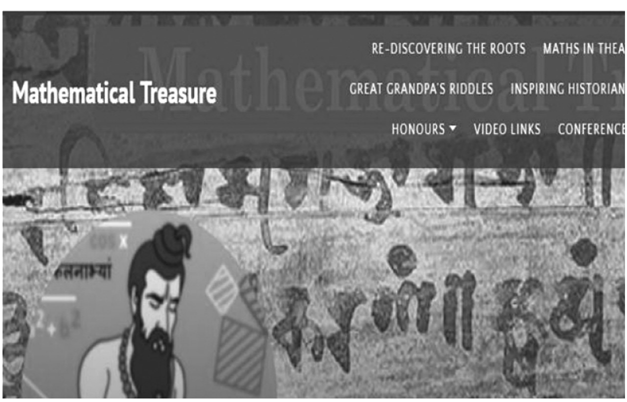


fig. 29.1: Main Tab 1

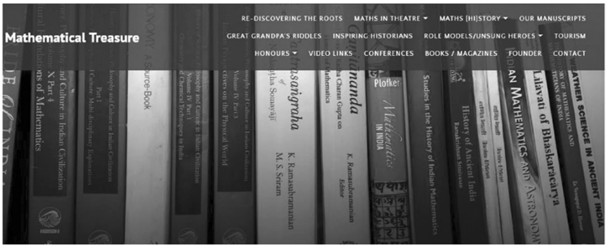


fig. 29.2: Main Tab 1

About Tab 1: Rediscovering the Roots

It starts with catchy headings like "Re-Discovering the Roots" and "Conserving our precious past for a marvellous future …".

fig. 29.3: Our Irreplaceable Heritage of Millenniums

Audio of a *śloka* has been included with the heading "The below music might strike a chord in you ...".

Then objective of the site is explained:

> … is to bring to the notice of today's youth about our rich mathematical heritage – the Indian heritage of innovating ideas, and of astonishingly advanced thoughts and the beautiful amalgamation of the arts and mathematics. Also it is proposed to pay homage to all those Indian mathematicians whose immense contributions to this universal subject have not been duly recognized. This knowledge of "History of Indian Mathematics" would lead the young minds to realize that mathematics is not only a subject, but also a part of their culture.

This tab will connect us to the 2nd tab "Maths in Theatre".

Here, about the beginning of this initiative is explained. I express my gratitude to my students and all who lent a helping hand in putting up this play. A short video, first 15 minutes of the play, is also uploaded. Excerpts from an article published in *The Hindu* newspaper of Hyderabad edition on 12 December 2000 with picture of media coverage in the background are put up.

This tab will connect us to the sub-tab, "Maths Drama: Journey through Maths – Crest of the Peacock", of main tab "Maths in Theatre".

In this sub-tab, gist of the content of the play is provided:

> The story starts from Indus Valley civilization, Vedic period, Jaina Mathematics, Bhakśāli Manuscript, Āryabhaṭa, Brahmagupta, Mahāvīrācārya, Bhāskarācārya, Story of Zero and Decimal System, Mādhavācārya & Mathematics from Kerala School and ends with tributes to Śrīnivāsa Rāmānujan (that way showcasing history of 5,000 years). A blend of folk narrative art form called "Burra Katha" of Andhra Pradesh and present-day technology is the medium of narration. This play includes five dance sequences, a few Sanskrit *śloka*s and about 80 slides.
>
> In ancient India, mathematics was not only a subject, but also it was part of the culture. The questions on maths used to be on birds, bees, animals, rivers and flowers. Maths was applied in temple architecture, music, *śrī yantra*s and magic squares. But, surprisingly, the subject was also quite advanced then. Calculus, a very important branch of mathematics has its origins in Kerala (fourteenth and seventeenth century).
>
> In history, one will come across the "so-called Pythagoras Theorem", "so-called Pascal triangle", "so-called Pells equation".
>
> It is quite wonderful to know how Trigonometric ratio "Sine" got its name.
>
> Duration of the play is approximately one hour. A team of 10 persons is performing.

Also, there is a call for the people who would like to promote this initiative.

> We are looking for individuals/institutions interested in promoting this project. Theatre is an effective tool to peep into India's glorious past and an opportunity to catch a few insights into mathematics and a few "values" as well.

Benefits of studying history of Indian mathematics are mentioned along with a few recommendations and aspirations.

This sub-tab will connect us to the sub-tab "August Audience". Background picture of audience has been provided.

The play was performed before:

1. Science educators and others of HBCSE, TIFR, Mumbai on 28 February 2000.
2. The scientists of Tata Institute of Fundamental Research, Mumbai on 18 April 2000 (World Heritage Day).
3. Delegates of International Conference on Statistics, (organized on the occasion of Professor C.R. Rao's 80th birthday celebrations), Hyderabad on 13 December 2000.
4. To the faculty of the University of Hyderabad on 13 December 2000.
5. Delegates of International Conference on History of Mathematical Sciences, Delhi in December 2002.
6. Students of different schools and colleges in Mumbai, Hyderabad and Delhi.
7. Students of St. Ann's School, Fort; Ruia College, K.C. College, Mumbai, and the students who attended National Science Day celebrations, etc.
8. A drama academy "Magic If" had taken this play as a project and performed in schools of Hyderabad for 50 times. Efforts of Mr Raj Shekhar, the director of the academy, are appreciated.

This sub-tab connects us to the "media coverage". Press coverage on my efforts over the years is on display in this.

Media sub-tab connects to the sub-tab "testimonials". A few of the testimonials are displayed below:

fig. 29.4

fig. 29.5

fig. 29.6

Tracing the history of modern mathematics

Staff Reporter 16, Feb, 2000

THE Mathematics department of St Xavier's College will stage a unique skit titled *The Crest of the Peacock*, which will take the audience on an exhilarating journey through the history of Indian mathematics. The skit will be staged by Xavierite students on February 16 at the college hall at 2 pm. The play is open to all.

A project done jointly by junior college lecturer P Sharad Devi and Dr Subramanyan from Homi Bhabha Centre for Science Education, the script of the play was written by Professor Devi along with SYBA student Ashok Balakrishnan. Year 2000 has been declared as the 'Year of Mathematics' by UNESCO.

The Crest of the Peacock will enlighten students about the rich heritage of ancient Indian mathematics which through innovation and ideas was astonishingly advanced compared to the West. The play starts in 3000 BC and narrates the achievements of mathematics till 2000 AD. "The play will lead students through maths during the Indus Valley era to the Vedic period, from Jain Mathematics to Bakshali maths, from Aryabhatta to Bramhagupta and the Kerala school of mathematics. The play will also have a slide show," said Devi.

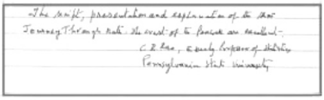
The play will show how the construction and design of the Indus Valley civilisation was as methodical as New York City, and how mathematics was involved in everything from the fire altars to the sacrificial vedis during the Vedic period. The play will also focus on the importance of Jain Mathematics which used it as religious roots to pave the way for abstract mathematical findings. Later, Bhakshali's manuscripts brought maths to a practical level and the concepts of profits and losses evolved.

The play will also feature the works of celebrated Indian mathematicians ranging from Aryabhatta to Bramhaputra and Bhaskaracharya. The findings of Aryabhatta, who is considered the father of advanced mathematics and the creator of the magical zero number, were later refined during the Bramhaputra period. It was during Bramhaputra's times that the glory of Indian mathematics spread across the world. The play will also touch upon the contribution of Bhaskaracharya and the enchanting tale of his treatise, *Lilavati*. The play will end with a glimpse of the Kerala school of mathematics, which developed the advanced calculus. "It is tragic but this treasure still remains hidden for many of us," said Devi.

bombaytimes@indiatimes.com

fig. 29.7

fig. 29.8



The script, presentation and explanation of the show Journey Through Math: The crest of the Peacock are excellent.
C R Rao, [illegible] Professor of Statistics
Pennsylvania State University

The script, presentation and explanation of the show "journey through maths – The crest of the peacock" are excellent.
– C.R. Rao, Pennsylvania State University

fig. 29.9

Dear Contributors of the documentary,
the play had been successfully enacted not only enacted but also for enlightening us about the glorious past. Thanks to the school of fine arts for having rendered their cooperation towards success. My sincere gratitude to the narrator and the director for their deep interest and the idea of implementing and staging it. The play is indeed one of rare ones of the modern times. Hope you would prepare such programs in future also. Wish you all grand success.
– T. Hema, BSc, St. Francis College

fig. 29.10

Testimonials sub-tab connects us to gallery.

fig. 29.11

fig. 29.12

This main tab concludes here.

Now we will move on to the third tab "Math (Hi)Story" to explore more. In the main tab, many video links are provided which enable the amateur in this to know more. A part of history of Indian mathematics is briefly narrated in six sub-tabs and by a main tab "Role Models/Unsung Heroes".

> The story of Indian mathematics which begins with Indus Valley Civilization, around 3000 BCE and come all the way to the twentieth century that's 5,000 years!!.
>
> The stages in this journey are:

Indus Valley (3500 BCE)	Mathematics in Vedic Saṁhitās (1750 BCE)
Śulbasūtras (600–200 BCE)	Maths in Ancient Jaina works (300 BCE to 200 CE)

Bhakśāli manuscript (300 CE)	Āryabhaṭa (476 CE)
Brahmagupta (598 CE)	Mahāvīracārya (ninth century)
Bhāskarācārya (1114 CE)	
Mādhavācārya and works	From Kerala (fourteenth to nineteenth century)
Śrīnivāsa Rāmānujan	(22 December 1887 – 26 April 1920)

In this journey, one would also examine the spread of zero and decimal system from India to Arabia and then to Europe.

The following additions are there in these sub-tabs:

a. An article in the form of a English lesson to 10th/11th standard presented by Sarada Devi on "Role models from our cultural roots" at Pune conference in 2014.
b. The letters between Hardy and Rāmānujan are presented in a poster form.

In the sub-tab "Unsung Heros", a small list of the names of the mathematicians from the ancient past is provided. This tab leads to main tab "Manuscripts".

This tab starts in the following way:

A leaf might contain huge wealth of knowledge such as a new branch of science, a method to prepare life-saving medicine … who knows what it can bestow on us.

India WAKE UP ... Conserve them, preserve them. Your heritage needs you.

India ... RISE AGAIN Uttiṣṭha … Bhārata

In this tab, the pathetic state of manuscript is briefly discussed. A monologue by a personified manuscript is provided. A few useful links are also given, e.g. https://namami.gov.in/

This tab connects us to the main tab "Books/Magazines". A small list of the books is provided. The contents of this tab will keep increasing in the future.

This main tab connects us to another main tab "Video Links". This tab contains rich resource material for the researchers as many video links by top historians are provided. This main tab connects us to another main tab "Conferences".

In this main tab "Conferences", I have included all the scanned copies of the certificates that I have attended on this subject (I believe 70 per cent of the such conferences I have attended). This tab is also under construction.

fig. 29.13

fig. 29.14

fig. 29.15

fig. 29.16

This main tab connects us to the another main tab "Inspiring Historians".

In this main tab, a few historians' picture slides are given with theme "Endaro Mahaanu Bhaavulu" (Many great souls). Only a small fraction is done here. This tab is under construction. Names of the various institutions which are working on the history of Indian maths will be provided in the future.

This tab takes us to "Honours Program" tab.

A department or a course in colleges and universities on this subject is highly required. A course either short or long is rare to find in India on the history of Indian mathematics. Efforts and contributions of P. Sarada Devi towards this cause have taken shape in the "Honours Program" at St. Xavier's College, Mumbai. The honours program was conducted thrice in St. Xavier's College, Mumbai, i.e. in 2004, 2007 and 2008.

The introduction, projects list, gallery and testimonials are given as sub-tabs.

fig. 29.17

fig. 29.18

Another interesting main tab is "Tourism" in mathematics. The places are Nīla River Banks, Ghāṭs and Saṅgama Grāma (due to Calculus), Caturbhuja Temple (due to zero), Jantar Mantar in Jaipur and Ujjain (due to Sun dial and *yantra*s), Pāṭana Devī in Chālīsgāon, Maharashtra (due to Bhāskarācārya), Chānd Bāoṛī of Rajasthan (due to symmetry), Inscription with Brāhmī numerals in Nānā Ghāṭ, Maharashtra, and Dholāvīrā, Indus Valley site in Gujarat (due to numerate culture) and Home of Śrīnivāsa Rāmānujan and Museum in Tamil Nadu. This list will also increase in the future.

fig. 29.19

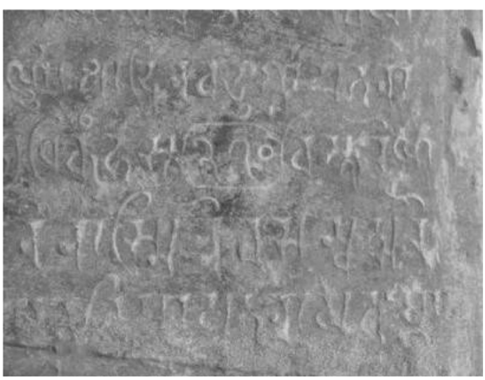


fig. 29.20

Another fascinating main tab is "Great Grandpa Riddles" or brain boosters.

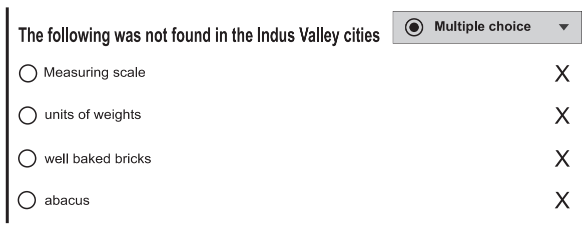
In this tab we have provided a few multiple choice questions on History of Indian Maths (HIM) and a few math questions from ancient texts like the *Līlāvatī*, Bhakśāli manuscript and *Gaṇitasāra-*

saṁgraha. We call them as "Great Grandpa Challenges". Students will enjoy solving them.

Example 1

At a distance of 200 cubits from a hill which is 1 cubits high is situated near a pond. Two hermits are at the top of the hill. One of them climbs down the hill and goes to the pond, while the other with his Yogic powers jumps up some distance into the air above the hill, and comes straight to the pond. Oh learned man! if you are well versed in mathematics, tell me how high did the second hermit jump into the sky, if the distance travelled by the both hermits are equal.

Example 2:


The following was not found in the Indus Valley cities — Multiple choice

- ○ Measuring scale X
- ○ units of weights X
- ○ well baked bricks X
- ○ abacus X

fig. 29.21

fig. 29.22

fig. 29.23

Finally, there is a main tab for the "Founder" of the website.

Another tab for the "Contact" details on the website.

Email: saradapattisapu@gmail.com

Phone : +91-9985851712

Conclusion

There is a lot of scope to expand this site. For example, I could have included a tab on "Curriculum". An online discussion is also a good idea. Also, I wish there will be many more websites by all the researchers/historians/educators on this subject. That will pave the way for the growth of this subject.

Acknowledgements

My gratitude to Ms. Aditya Vadlapudi for designing this website and for her valuable suggestions and to Ms. Harshitha Senapathi for suggestions and proofreading.

References

Bose, D.M., S.D. Sen and B.V. Subbarayappa, 1971, *A Concise History of Science in India*, New Delhi: Indian National Science Academy.

Chattopadhyaya, D.P., 1986, *History of Science and Technology in Ancient India*, Calcutta: Firma KLM.

Datta, B. and A.N. Singh, 1962, *History of Hindu Mathematics*, 2 vols, Bombay: Asia Publishing House.

Dunham, William, 1997, *The Mathematical Universe*: *An Alphabetical Journey through the Great Proots, Problems and Personalities*, New York: Wiley & Sons.

Jaggi, O.P., 1929, *Concise Hisotory of Science in India*, Delhi: Atma Ram.

Joseph, George Gheverghese, 1995, *The Crest of the Peacock*: *Non European Roots of Mathematics*, New Jersey: Princeton University Press.

Patwardhan, Krishnaji Shankara, Somashekhara Amrita Naimpally, Shyam Lal Singh, 2001, *Lilavati of Bhaskaracharya*: *A Treatise of Mathematics of Vedic Tradition*, Delhi: Motilal Banarasidass.

Saraswati, T.A., 1979, *Geometry in Ancient and Medieval India,* Delhi: Motilal Banarsidass.

Satya Prakash, 1965, *Founders of Sciences in Ancient India*, New Delhi: The Research Institute of Ancient Scientific Studies.

Srinivasa Iyengar, C.N., 1967, *The History of Indian Mathematics*, Calcutta: World Press.

Subbarayappa, B.V. and N. Mukunda (eds), 1995, *Science in the West and India: Some Historical Aspects*, New Delhi: Hindustan Publishing House.

Cultural Heritage Series, vol. 6, Bharatiya Vidya Bhavan.

Ganita Bharati, a magazine, ISHM.

30

Technology of Veda Mantra Transmission through Ages

Relevancy of Current Communication Technology (Verbal and Text)

M. Rajendran

THE oral tradition of Vedic chanting has been declared an intangible heritage of humanity by UNESCO. In a meeting of jury members on 7 November 2003 in Paris, Koichiro Matsuura, Director General of UNESCO, declared the chanting of Vedas in India as an outstanding example of heritage and the form of cultural expressions. The proclamation says in the age of globalization and modernization that when the cultural diversity is under pressure, the preservation of oral tradition of Vedic chanting, a unique cultural heritage, has great significance.

Divisions of the Four Vedas

The Veda is considered to be infinite (*ananto vai vedāḥ*). In the beginning of creation there was only one *veda* and the number of revealed texts was far greater than we could imagine, during the course of time due to the diminishing intelligence of mankind as well as its declining strength, health and loss of faith, many texts were lost and the *veda* that is known today is a mere fraction of the original *veda*.

Towards the close of the Dvāpara-Yuga, it is believed, the Lord manifested as Sage Veda Vyāsa, who in order to save the *veda* from extinction, re-edited it, dividing it into four units. Each unit was assigned to different classes of brāhmaṇas so that it would be easier to preserve them. These four units are known as the *Ṛk*, *Yajur*, *Sāma* and *Atharva*.

Veda Vyāsa had four disciples and to each of them he taught one *veda*. Paila mastered the *Ṛgveda*, Jaimini the *Sāmaveda*, Vaiśampāyana the *Yajurveda* and the *Atharvaveda* was learnt by Sumantu. Romaharṣana was entrusted with the duty of transmitting the Purāṇas and Itihāsas.

The Vedas transmitted by these sages to their disciples and in turn by the latter to theirs resulted in the Vedas becoming diversified into many branches or schools through the disciplic succession.

Vedic Chant

The Vedic chant is the oldest form of psalmody known. Very strict and complex methods of instruction have made it possible to preserve the ritual chant unchanged, despite thousands of years of wars, conquests and social upheavals. The *Ṛgveda* is chanted on three notes, the *Yajurveda* on up to five notes and the *Sāmaveda* on seven notes. The *Sāma* is the only chant that is considered really musical per se and as such is considered to be inferior to the other two Vedas. Because of its "worldly" character it is often forbidden in certain rituals. It is also prescribed that if the *Sāmaveda* is heard while the other two are being recited then the recitation should stop immediately and only continue after the *Sāma* has terminated. According to the *Taittirīya Upaniṣad* – Śīkṣā-vaḷḷī – there are six main factors that need to be taken into consideration.

i. *Varṇaḥ* – pronunciation
ii. *Svaraḥ* – notes
iii. *Mātrā* – duration
iv. *Balaṁ* – emphasis
v. *Sāma* – Uniformity
vi. *Santānaḥ* – Continuity

The Vedic Accent

The rules of correct pronunciation and articulation of sounds are given in the Vedāṅga, known as *śīkṣā*.

> शीक्षा व्याख्यास्यामः। वर्णस्वरः। मात्रा बलं। साम सन्तानः। इत्युक्त शीक्षाद्यायः।।
>
> *Śīkṣā* deals with *varṇa* (letters); *svaraḥ* (pitch) [there are essentially three *svaras*, viz. *anudātta* (gravely accented or low-pitched), *udātta* (high-pitched or acutely accented), *svarita* (circumflexly accented)]; *mātrā* (duration – a prosodial unit of time); *balaṁ* (strength or force of articulation); *sāma* (uniformity); and *santānaḥ* (continuity) during recitation.

Variant Forms of Vedic Chant

Vedic recitation has assumed two distinct forms that evolved to preserve its immutable character: *prakr̥ti* and *vikr̥ti* with sub-forms.

The *pāda-pāṭhaḥ* forms the basis of a number of special recitations known as *vikr̥ti* (crooked) recitations. The text is recited backward or forward or the successive words are chanted in specific combinations. These were originally designed to prevent the student from forgetting even one letter of the text, however, through the ages, these mnemonic techniques became an end in themselves.

PRAKR̥TI

Saṁhitā-pāṭhaḥ

ओषधयः सं वंदन्ते सोमेंन सह राज्ञीं।

Pāda-pāṭhaḥ

ओषंधयः। सं। वदन्ते। सोमेंन। सह। राज्ञीं।।

Krama-pāṭhaḥ

ओषंधयः। सं। सं वंदन्ते। वदन्ते सोमेंन। सोमेंन सह। सह राज्ञीं। राज्ञेति राज्ञीं।।

Mathematical Sequence Series of Krama-pāṭhaḥ

Sentence (S) = $P1, P2, P3, \ldots, P(n-2), P(n-1), Pn$

Krama Turn T, number ($T1$ to $n - 1$)

Turn 1 ($T1$) = $P1$, $P2$;

Turn 2 ($T2$) = $P2$, $P3$;

Turn ($n - 2$) = $P(n - 2)$, $P(n - 1)$

Turn ($n - 1$) = $P(n - 1)$, $P(n)$

General combination for *krama-pāṭhaḥ* is

Turn ($n - 1$) $T(n - 1) = P(n - 1)$, $P(n)$;

where, $n > 1$ and maximum number of turns $< n$ (without any *veṣṭana*)

$Pn = n^{\text{th}}$ *pāda* in the sentence

Tn = Turn of *krama-pāṭhaḥ*

n = Number of *pāda* in a sentence

In the *prakṛti* form, the words do not change their sequence.

VIKṚTI

The *vikṛti*s are given in the following verse:

जटा माला शिखा रेखा ध्वजो दण्डो रथोघनः। इत्यष्टा विकृतयः प्रोक्ताः क्रमपूर्वा महर्षिभिः

1. *jaṭā*; 1 2 2 1 1 2 / 2 3 3 2 2 3 / 3 4 4 3 3 4 / 4 5 5 4 4 5 / ...

ओषधयः सं सं ओषधयः ओषधयः सम्। सं वदन्ते वदन्ते सं सं वदन्ते। वदन्ते सोमेन सोमेन वदन्ते वदन्ते सोमेन। सोमेन सह सह सोमेन सोमेन सह। सह राज्ञा राज्ञी सह सह राज्ञी। राज्ञेति राज्ञी।।

Mathematical Sequence Series of Jaṭā-pāṭhaḥ

Sentence (S) = $P1, P2, \ldots, P(n - 2), P(n - 1), P(n)$

Jaṭā Turn (T), Number ($T1$ to $n - 1$)

Turn 1 ($T1$) = $P1$, $P2$, $P2$, $P1$, $P1$, $P2$

Turn 2 ($T2$) = $P2$, $P3$, $P3$, $P2$, $P2$, $P3$

Turn ($n - 2$)$T(n - 2) = P(n - 2), P(n - 1), P(n - 1), P(n - 2), P(n - 2), P(n - 1)$

Turn $(n - 1)(T(n - 1)) = P(n - 1), P(\mathrm{n}), P(n), P(n - 1), P(n - 1), P(n)$

General combination of *jaṭā-paṭhaḥ* is

Turn $(n-1)(T(n-1)) = P(n-1), P(n), P(n), P(n-1), P(n-1), P(n)$

where, $n > 1$ and maximum number of turns $< n$ (without any *veṣṭana*)

$Pn = n^{\text{th}}$ *pāda* in the sentence

Tn = Turn of *jaṭā-pāṭhaḥ*

n = Number of *pāda* in a sentence

2. *mālā*; 1 2 / 2 1 / 1 2 / 2 3 / 3 2 / 2 3 / 3 4 / 4 3 / 3 4 / ...
3. *śikhā*; 1 2 2 1 1 2 3 / 2 3 3 2 2 3 4 / 3 4 4 3 3 4 5 / 4 5 5 4 4 5 6/ ...
4. *rekhā*; 1 2 / 2 1 / 1 2 / 2 3 4 / 4 3 2 / 2 3 / 3 4 5 6 / 6 5 3 4 / 3 4 /4 5 6 7 8 / 8 7 6 5 4 / 4 5 / 5 6 7 8 9 10 / 10 9 8 7 6 5 / 5 6 / ...
5. *dhvaja*; 1 2 / 99 100 / 2 3 / 98 99 / 3 4 / 97 98 / 4 5 / 97 98 / 5 6 / 96 97 / ... 97 98 / 3 4 / 98 99 / 2 3 / 99 100 / 1 2 .
6. *daṇḍa*; 1 2 / 2 1 / 1 2 / 2 3 /3 2 1 / 1 2 / 2 3 / 3 4 / 4 3 2 1 / 1 2 / 2 3 / 3 4 / 4 5 / 5 4 3 2 1 ...
7. *ratha*; 1 2 / 5 6 / 2 1 / 6 5 / 1 2 / 5 6 / 2 3 / 6 7 / 3 2 1 / 7 6 5 / 1 2 / 5 6 / 2 3 / 6 7 / 3 4 / 7 8 / 4 3 2 1 / 8 7 6 5 / ...
8. *ghana*; 1 2 2 1 1 2 3 3 2 1 1 2 3 / 2 3 3 2 2 3 4 4 3 2 2 3 4 / 3 4 4 3 3 4 5 5 4 3 3 4 5 / ...

ओषधयः सं सं ओषधयः ओषधयः सं वंदन्ते वदन्ते सं ओषधयः ओषधयः सं वदन्ते। सं वंदन्ते वदन्ते सं सं वंदन्ते सोमेंन सोमेंन। वदन्ते सं सं वदन्ते सोमेंन। वदन्ते सोमेंन सोमेंन वदन्ते वदन्ते सोमेंन सह सह सोमेंन वदन्ते वदन्ते सोमेंन सह। सोमेंन सह सह सोमेंन सोमेंन सह राज्ञीं सह राज्ञा राज्ञीं सह सह राज्ञीं। राज्ञेति राज्ञीं।।

Mathematical Sequence Series of Ghana-pāṭhaḥ

Sentence (S) = $P1, P2, \ldots, P(n-2), P(n-1), P(n)$

ghana Turn (T), Number ($T1$ to $n-1$)

Turn1 ($T1$) = $P1, P2, P2, P1, P1, P2, P3, P3, P2, P1, P1, P2, P3$

Turn 2 ($T2$) = $P2, P3, P3, P2, P2, P3, P4, P4, P3, P2, P2, P3, P4$

Turn $(n-2)\mathrm{T}(n-2) = P(n-2), P(n-1), P(n-1), P(n-2),$

$$P(n-2), P(n-1), Pn, Pn, P(n-1), P(n-2), P(n-2), P(n-1), Pn$$

$$\text{Turn } (n-1)(T(n-1)) = P(n-1), P(n), P(n), P(n-1), P(n-1), P(n)$$

General combination of *ghana-pāṭhaḥ* is

$$\text{Turn } (n-1)(T(n-1)) = P(n-2), \mathrm{P}(n-1), P(n-1), P(n-2), P(n-2),$$
$$P(n-1), P(n), P(n), P(n-1), P(n-2), P(n-2), P(n-1), P(n),$$

where, $n > 1$ and maximum number of turns $< n$ (without any *veṣṭana*)

$Pn = n^{\text{th}}$ *pāda* in the sentence

Tn = Turn of *ghana-pāṭhaḥ*

n = Number of *pāda* in a sentence.

CHANDAS (METRE)

The metres are regulated by the number of syllables (*akṣara*s) in the stanza (*ṛk*), which consists generally of three or four *pāda*s, measures, divisions, or quarter verses, with a distinctly marked interval at the end of the second *pāda*, and so forming two semi-stanzas of varying length.

The most common metres consisting of 8, 9, 10, 11, 12 syllables (*akṣara*s) in each *pāda*, are known as Anuṣṭubh, Bṛhatī, Paṅkti, Triṣṭup and Jagatī.

The Anuṣṭubh is the prevailing form of metre in the Dharma-śāstras, the *Rāmāyaṇa*, the *Mahābhārata* and all the Purāṇas.

The *pāda*s of a stanza are generally of equal length and of more or less corresponding prosodial quantities. But, sometimes two or more kinds of metre are employed in one stanza; then the *pāda*s vary in quantity and length.

Maharṣi Piṅgala Chandasūtram

"Maharṣi Piṅgala Chandasūtram and Computer Binary Algorithms" is an unusual topic which links the past and the present. Computers represent the modern era, the Vedas are of a hoary past. Much has been researched and documented about computers, the Vedas are still to be solved of their mysteries. Many Vedic hymns have astounded the modern scientists and astronomers, but there has been no serious effort to unravel the real meanings behind all the Vedic hymns. Here, we present the relevant binary system *sūtra*s with the explanation and working of the algorithms written in coded *sūtra*s. This opens up new areas for research and implementation of Piṅgala's left to right binary or the Big-endian system.

Of the various gifts the Hindus gave to the world, the knowledge of *gaṇita* (mathematics) is supreme. They gave the concept of *śūnya* (zero), the decimal system (base 10) and sexadecimal (base 60) system. The binary system, which forms the basis of computation and calculation in computers, seems to be the superlative discovery of modern mathematics. It is astonishing to find the binary system in the Vedāṅga of *chanda*s given so clearly by Maharṣi Piṅgala. As with any ancient Vedic knowledge, the binary system has been hidden in the *Chandasūtram*. The Hindus' unique method is of using Sanskrit *akṣara*s (alphabets) for writing numbers left to right, with the place value increasing to the right. These are read in the reverse order from right to left – *aṅkanam vāmato gatiḥ*. The binary numbers are also written in the same manner as decimal numbers and read from right to left. We present the relevant *sūtra*s from Maharṣi Piṅgala's *Chandasūtram*. The algorithms are written as *sūtra*s. The algorithms are recursive in nature, a very high concept in modern computer programming language. We fix the date of this Vedāṅga based on the date of the Vedas.

Very large numbers have been encoded using the algebraic code of Maharṣi Piṅgala's *Chandasūtram*. The conformity between decimal and binary number is given in the *Adhvayoga*. This has to be properly understood, these *akṣara* binary numbers are not used for enumeration and classification of *chanda*s only. *Chanda* means covering, hiding or concealing according to Vedic etymology. According to Pāṇini, it means Vedas and Vedic language. Prastara gives the algorithm for changing an ordinal number to *guru–laghu* binary syllabic encoding. Similar is the scheme of *kaṭapayādi* changing numbers to meaningful mnemonics. (We have developed software programs of the algorithms given in Maharṣi Piṅgala's *Chandasūtram*). The algorithms should have been formulated before the specific Veda *mantra*s. And the *Vedāṅga-Jyotiṣa* gives the algorithms for astronomical calculations. To memorize the large volume of astronomical data and calculation tables Maharṣi Piṅgala's binary system was used. This astronomical calculations were necessary for making rituals at appropriate time as given in the *Kalpasūtras*. Maharṣi Piṅgala defined two series of numbers, index or serial number and a quantitative series. The quantitative series lists the meteric variations, and index number gives decimal values of the variations as per *adhvayoga* algorithm. The main purpose of the *Chandasūtram* is to give rules based on *bīja-gaṇita* (algebra) for encoding the *gaṇa*s or *akṣara* combinations.

*Chanda*s are for the study of Vedic metre. This gives the importance of "Encoding of the Veda Mantras". This is the *pāda* (foot) of the Vedas. This gives the cryptic astronomical, algebraical, geometrical and method of Vedic interpretation. This has been in use in Tamil grammar *Tolkāppiyam*. The science of metrics in Tamil is named as *Yappilakanam*. Almost all the technical terms of *Chandasūtram* have similar word-meaning in Tamil.

Interpretation of Vedas, based on the encoding methods using *Chandasūtram*, gives a method of chanting supercomputer. The *mantra*s are based on sound and not on written scripts. The duration of pronunciation, the rules for when a *laghu* (short vowel) is to be pronounced as *guru* (long vowel) gives the superiority of sound over script. And this forms the basis of committing to memory large numbers of astronomy using the coding schemes of *chanda*s.

Vedas are in different *chanda*s. One meaning of *chanda*s is that it is knowledge which is to be guarded in secret and propagated with care. The Vedas are also described as *chanda*s. The whole of *Sāmaveda* is consisted of *chanda*s. There is a word in Tamil referring to Tamil language as *chandahtamil*. Of the six Vedāṅgas, Chandaśāstra forms a part essential to understand the Vedas.

The following algorithms are for the binary system in Piṅgala's *Chandasūtram*.

Chandasūtram by Maharṣi Piṅgala contains eighteen *pariccheda* (sub-chapters) in eight *adhyāya*s (main chapters). The 1st *pariccheda* of six *śloka*s are not *sūtra*s. The rest of the *Chandasūtram* is composed of *sūtra*s.

The fourth *śloka* is:

mā ya rā sa tā ja bhā na la ga sammitam bhramati vaṅgamayam jagatiyasya |
sajayati piṅgala nāgaḥ śiva prasādat viśuddha matiḥ ||

And the sixth *śloka* is:

tri vīramam das varṇam ṣaṇmātramuacha piṅgala sūtram |
chandovarga padarta pratyaya hetoścasastaradou ||

In this Maharṣi Piṅgala states that *mā, ya, rā, sa, tā, ja, bhā, na, la, ga* mentioned in the fourth *śloka* is in itself a *sūtra*, containing ten *varṇa*s and specifies that the same is kept on the top of all *sūtra*s because it is the basis for *chando varga padārtha*s and *pratyaya*s. Three technical terms are given here: *vīramam*, *mātrā* and *pratyaya*. The term *pratyaya* indicates vast and remarkable meaning. The astonishing wonderful intelligence of Maharṣi Piṅgala is imbibed in various *pratyaya*s. In fact, the *pratyaya*s is a collection of extraordinarily ingenious and clever solutions to problems.

The 8th *adhyāya* gives the following sixteen *sūtra*s (8.20-8.35) which relate to the Piṅgala *pratyaya* system:

1. *Prastāraḥ* – Algorithms to produce all possible combinations of *n* binary digits.
2. *Naṣṭam* – Algorithms to recover the missing row.
3. *Uddiṣṭam* – Algorithms to get the row index of a given row.
4. *Saṁkhyā* – Algorithms to get the total number of *n* bit combinations.
5. *Adhvayoga* – Algorithms to compute the total combinations of *chanda*s ranging from 1 syllable to *n* syllables.
6. *Eka-dvi-adi-l-g-kriyā* – Algorithms to compute a number of combinations using *n* – number of syllables taking *r* – the number of *laghu*s (or *guru*s), at a time *nCr*.

Conclusion

Interpretation of Vedas based on the encoding methods using *Chandasūtram* gives a method of chanting supercomputer. The *mantra*s are based on sound and not on written scripts. The duration of pronunciation, the rules for when a *laghu* (short vowel) is to be pronounced as a *guru* (long vowel) gives the superiority of sound over script. And this forms the basis of committing to memory large numbers of astronomy using the coding schemes of *chanda*s. Vedas are in different *chanda*s (metres). One meaning of *chanda*s is that it is knowledge which is to be guarded in secret and propagated with care. The Vedas are also described as chandas. The whole of Samaveda consists of *chanda*s. There is word in Tamil referring Tamil language as Chandahtamil. Of the six Vedaṅgas Chandasāstra forms a part essential to understand the Vedas. These *chanda*s have been studied in great details. Vikrutti's or chanting method serves the purpose of retaining intact in veda *mantra*s without any error throughout the ages. Pingala *chanda*s give the rules for encoding knowledge inside the Veda *mantra*s. These systems have to be researched and adapted for currently communication technology.

References

Chanda Sutra by Maharishi Pingala

https://en.wikipedia.org/wiki/Sanskrit_prosody

https://veda.wikidot.com/tip:vedic-chanting

Vedic Chant – https://www.rkmkhar.org/spiritual-activities/vedic-chanting/

31

A Note on Confusion Matrix and Its Real Life Application

T.N. Kavitha

Abstract: A discussion of the origin of the confusion matrix and a variety of definition of various persons are given in a detailed manner. A confusion matrix contains information about actual and predicted classifications done by a classification system. Performance of such systems is commonly evaluated using the data in the matrix. The proportion of a data set for which a classifier makes a prediction. If a classifier does not classify all the instances, it may be important to know its performance on the set of cases for which it is "confident" enough to make a prediction, that matter is discussed herein in a detailed manner.

Keywords: Confusion matrix, classifier, prediction, contingency table.

Introduction

A CONFUSION matrix, also known as an error matrix, is a specific table layout that allows visualization of the performance of an algorithm, typically a supervised learning one (in unsupervised learning, it is usually called a matching matrix). Each row of the matrix represents the instances in a predicted class while each column represents the instances in an actual class (or vice versa)

(Pearson 1904). The name stems from the fact that it makes it easy to see if the system is confusing two classes (i.e. commonly mislabelled one as another). It is a special kind of contingency table, with two dimensions ("actual" and "predicted"), and identical sets of "classes" in both dimensions (each combination of dimension and class is a variable in the contingency table).

Meaning of Confusion Matrix

In *Oxford Dictionary of Psychology*, we have the following definition for confusion matrix:

> A matrix representing the relative frequencies with which each of a number of stimuli is mistaken for each of the others by a person in a task requiring recognition or identification of stimuli. Analysis of these data allows a researcher to extract factors (2) indicating the underlying dimensions of similarity in the perception of the respondent. For example, in colour-identification tasks, relatively frequent confusion of reds with greens would tend to suggest daltonism. – Matthew

The confusion matrix was invented in 1904 by Karl Pearson. He used the term Contingency Table. It appeared at Karl Pearson's *Mathematical Contributions to the Theory of Evolution*. During the Second War World, detection theory was developed as an investigation of the relations between stimulus and response. We have used confusion matrix there. Due to the detection theory, the term was used in psychology. From there the term reached machine learning. In statistics, it seems that though the concept was invented, a field very related to the machine learning, it reached machine learning after a detour in during 100 years.

J.T. Townsend introduced the concept of confusion matrix in his paper "Theoretical Analysis of an Alphabetic Confusion Matrix" (1971). In this work, a study was undertaken to obtain a confusion matrix of the complete upper-case English alphabet with a simple non-serifed font under tachistoscopic conditions. This was accomplished with two experimental conditions, one with blank post-stimulus field and one with the noisy post-stimulus field, for six (sensory states) Ss run 650 trials each. Three mathematical models of recognition, two based on the concept of a finite number of sensory states and one being the choice model were compared in their ability to predict the confusion matrix after their parameters were estimated from functions of the data.

The paper discusses an experiment in which the 26 English alphabet letters (stimuli) are presented to a subject that should present a reply with the same letter (reaction). The confusion is a 26 × 26 matrix with the probability of each reaction to each stimulus. This explains the name (the matrix of the subject confusion) and matches the use in machine learning today.

Ron Kohavi and Foster Provost discussed about confusion matrix in the topic "Glossary of Terms" (1998).

They defined a matrix called confusion matrix showing the predicted and actual classifications. A confusion matrix is of size L x L, where L is the number of different label values. The following confusion matrix is for L = 2:

Actual\Predicted	*Negative*	*Positive*
Negative	A	B
Positive	C	D

The following terms are defined for a two × two confusion matrix:

Accuracy: $(a + d)/(a + b + c + d)$.

True positive rate (recall, sensitivity): $d/(c + d)$.

True negative rate (specificity): $a/(a + b)$.

Precision: $d/(b + d)$.

False positive rate: $b/(a + b)$.

False negative rate: $c/(c + d)$.

Coverage

The proportion of a data set for which a classifier makes a prediction. If a classifier does not classify all the instances, it may be important to know its performance on the set of cases for which it is "confident" enough to make a prediction.

The very first Howard Hamilton described this concept in his 2002 article named "Confusion Matrix".

A confusion matrix (Kohavi and Provost 1998) contains information about actual and predicted classifications done by a classification system. Performance of such systems is commonly evaluated using the data in the matrix. The following table shows the confusion matrix for a two class classifier.

		Predicted	
		Negative	*Positive*
Actual	Negative	*a*	*B*
	Positive	*c*	*D*

The entries in the confusion matrix have the following meaning in the context of our study:

- *a* is the number of "correct" predictions that an instance is negative,
- *b* is the number of "incorrect" predictions that an instance is "positive",
- *c* is the number of "incorrect" of predictions that an instance "negative", and
- *d* is the number of "correct" predictions that an instance is "positive".

Several standard terms have been defined for the two class matrix:

- The *accuracy* (*AC*) is the proportion of the total number of predictions that were correct. It is determined using the equation:

$$AC = \frac{a+d}{a+b+c+d}. \tag{1}$$

- The *recall or true positive* (*TP*) rate is the proportion of positive cases that were correctly identified, as calculated using the equation:

$$TP = \frac{d}{c+d}. \tag{2}$$

- The *false positive* (*FP*) rate is the proportion of negatives cases that were incorrectly classified as positive, as calculated using the equation:

$$FP = \frac{b}{a+b}. \tag{3}$$

- The *true negative* (*TN*) rate is defined as the proportion of negatives cases that were classified correctly, as calculated using the equation:

$$TN = \frac{a}{a+b}. \tag{4}$$

- The *false negative* (*FN*) rate is the proportion of positives cases that were incorrectly classified as negative, as calculated using the equation:

$$FN = \frac{c}{c+d}. \tag{5}$$

- Finally, *precision* (*P*) is the proportion of the predicted positive cases that were correct, as calculated using the equation:

$$P = \frac{d}{b+d}. \tag{6}$$

The accuracy determined using equation (1) may not be an adequate performance measure when the number of negative cases is much greater than the number of positive cases (Kubat et al. 1998). Suppose there are 1,000 cases 995 of which are negative cases and 5 of which are positive cases. If the system classifies them all as negative, the accuracy would be 99.5 per cent, even though the classifier missed all positive cases. Other performance measures account for this by including *TP* in a product: for example, geometric mean (*g*-mean)(Kubat et al. 1998), as defined in equations (7) and (8) and *F*-measure (Lewis and Gale 1994), as defined in equation (9).

$$g\text{-mean}_1 = \sqrt{TP \times P}. \tag{7}$$

$$g\text{-mean}_2 = \sqrt{TP \times TN}. \tag{8}$$

$$F = \frac{(\beta^2 + 1) \times P \times TP}{\beta^2 \times P + TP}. \tag{9}$$

In equation (9), β has a value from 0 to infinity and is used to control the weight assigned to *TP* and *P*. Any classifier evaluated using equations 7, 8 or 9 will have a measure value of 0, if all positive cases are classified incorrectly.

Tom Fawcett published the topic "An Introduction to ROC Analysis" (2006). The matter discussed in this article is publication a review. Given a classifier and an instance, there are four possible outcomes.

1. If the case is positive and it is classified as positive, it is counted as a true positive.
2. If it is classified as negative, it is calculated as a *false negative*.
3. If the instance is negative and it is classified as negative, it is to add up as a *true negative*.
4. If it is classified as positive, it is counted as a *false positive*.

Given a classifier and a set of instances (the test set), a two × two "confusion matrix" (also called a "contingency table") can be constructed representing the dispositions of the set of instances. This matrix forms the basis for many common metrics.

Gregory Griffin, Alex Holub and Pietro Perona presented their effort about the confusion matrix named "Caltech-256 Object Category Dataset" (2007).

Kai Ming Ting presented in the similar way like the existing one, that is the attempt of 'Confusion Matrix' (2011).

In 2018, the following are very clear, i.e. in the field of machine learning and specifically the problem of statistical classification, a confusion matrix, also known as an error matrix, is a specific table layout that allows visualization of the performance of an algorithm, typically a supervised learning one. Each row of the matrix represents the instances in a predicted class while each column represents the instances in an actual class (or vice versa).

It is a special kind of contingency table, with two dimensions ("actual" and "predicted"), and identical sets of "classes" in both dimensions.

Example

A "confusion matrix" for a classification task with the three (c = 3) output classes: *A*, *B* and *C*. The test set used to evaluate the algorithm contained 100 cases with a distribution of 30 *A*s, 35 *B*s and 35 *C*s. A perfect classifier would have only made predictions along the diagonal, but the results below show that the algorithm was only correct on (20 + 25 + 24)/100 = 69 per cent of the cases. The "matrix" can be used to infer that the classifier often confuses dairy for cans (11 incorrect) and cans for dairy (9 wrong). This "matrix" also includes summations of the rows and columns.

ACTUAL/ PREDICTED	*A*	*B*	*C*	*sum*
A	20	2	11	33
B	2	25	1	28
C	9	5	24	38
Sum	**31**	**32**	**36**	**100**

Conclusion

A "confusion matrix" is a table that often used to describe the performance of a classification model (or "classifier") on a set of test data for which the true values are known. A confusion matrix is a contingency table that represents the count of a classifier's class predictions with respect to the actual outcome on some labelled learning set. Predictions areas were the function encounters with all its difficulties. The application of the confusion matrix allows the visualization of the performance of an algorithm in Python software.

References

Fawcett, Tom, 2006, "An Introduction to ROC Analysis", *Pattern Recognition Letters*, 27(8): 861-74.

Griffin, Gregory, Alex Holub and Pietro Perona, 2007, "Caltech-256 Object Category, Dataset", California Institute of Technology - Technical Report (unpublished).

Hamilton, Howard, 2002, "Confusion Matrix", *Class Notes for Computer Science 831: Knowledge Discovery in Databases*, University of Regina.

Kohavi, Ron and Foster Provost, 1998, "Glossary of Terms", *Machine Learning*, 30(2/3): 271-74.

Kubat, M., Holte, R., & Matwin, S. 1998, "Machine Learning for the Detection of Oil Spills in Satellite Radar Images", *Machine Learning*, 30: 195-215.

Lewis and Gale, 1994, *A Sequential Algorithm for Training Text Classifiers*, Annual ACM conference on Research and Development in Information Retrieval, the 17th Annual International ACM SIGIR Conference on Research and Development in Information Retrieval, pp. 3-12, New York: Springer-Verlag.

Pearson, Karl, 1895, *Contributions to the Mathematical Theory of Evolution II*, Skew Variation in Homogeneous Material, *Philosophical Transactions of the Royal Society of London*, *A* vol. 186, pp. 343-414, London: Royal Society Stable.

Pearson, Karl F.R.S., 1904, *Mathematical Contributions to the Theory of Evolution*, London: Dulau and Co.

Ting, Kai Ming, 2011, "Confusion Matrix", in *Encyclopedia of Machine Learning*, ed. Claude Sammut and Geopprey Web, p. 209, Boston: Springer.

Townsend, J.T., 1971, "Theoretical Analysis of an Alphabetic Confusion Matrix", *Perception & Psychophysics*, 9(1): 40-50.

Wikipedia, 2018 ⇒ https://en.wikipedia.org/wiki/confusion_matrix

http://ww2.cs.uregina.ca/~dbd/cs831/notes/confusion_matrix/confusion_matrix.html

32

Historical Development of Fluid Dynamics

E. Geetha
M. Larani

Abstract: In this paper we discuss about the history and development of fluid dynamics. Fluid dynamics is the subfield of fluid mechanics. Fluid mechanics is the combination of hydraulics and hydrodynamics. Hydraulics developed as an empirical science beginning from the pre-historical times. The advent of hydrodynamics, which tackles fluid movement theoretically, was in eighteenth century by various scientists. Complete theoretical equations for the flow of non-viscous fluid were derived by Euler and other scientists. In the nineteenth century, hydrodynamics advanced sufficiently to derivate the equation for the motion of a viscous fluid by Navier and Stokes: only laminar flow between parallel plates was solved. In the present age, with the progress in computers and numerical techniques in hydrodynamics, it is now possible to obtain numerical solutions of Navier–Stokes equation.

Keywords: Pascal's law, hydrostatics, hydrodynamics, Hagen–Poiseuille equation, Vortex Dynamics.

Introduction

THE history of fluid mechanics, the study of how fluids move and the forces on them, dates back to the ancient Greeks. A pragmatic, if not scientific, knowledge of fluid flow was exhibited by ancient civilizations, such as in the design of arrows, spears, boats and particularly hydraulic engineering projects for flood protection, irrigation, drainage and water supply (Garbrecht 1987). The earliest human civilizations began near the shores of rivers, and consequently, coincided with the dawn of hydrology, hydraulics and hydraulic engineering.

Archimedes

The fundamental principles of hydrostatics and dynamics were given by Archimedes in his work on floating bodies (ancient Greek), around 250 BCE. In it, Archimedes develops the laws of buoyancy, also known as Archimedes' Principle. This principle states that a body immersed in a fluid experiences a buoyant force equal to the weight of the fluid it displaces (Caroll 2007). Archimedes mentioned that each particle of a fluid mass, when in equilibrium, is equally pressed in every direction; and he inquired into the conditions according to which a solid body floating in a fluid should assume and preserve a position of equilibrium (Greenhill 1912) .

The Alexandrian

In the Greek school at Alexandria, which flourished under the auspices of the Ptolemies, attempts were made at the construction of hydraulic machinery, and in about 120 BCE the fountain of compression, the siphon and the forcing-pump were invented by Ctesibius and Hero. The siphon is a simple instrument; but the forcing-pump is a complicated invention, which could scarcely have been expected in the infancy of hydraulics. It was probably suggested to Ctesibius by the Egyptian wheel or Noria, which was common at that time, and which was a kind of chain pump, consisting of a number of earthen pots carried round by a wheel. In some of these machines the pots have a value in the bottom which enables them to descend without much resistance, and diminishes greatly the load upon the wheel; and, if we suppose that this value was introduced so early as the time of Ctesibius, it is not difficult to perceive how such a machine might have led to the invention of the forcing-pump (Greenhill 1911).

Sextus Julius Frontinus

Notwithstanding these inventions of the Alexandrian school, its attention does not seem to have been directed to the motion of fluids; and the first attempt to investigate this subject was made by Sextus Julius Frontinus, inspector of the public fountains at Rome in the reigns of Nerva and Trajan. In his work *De aquaeductibus urbis Romae* commentaries, he considers the methods which were at that time employed for ascertaining the quantity of water discharged from tubes and the mode of distributing the waters of a water supply or a fountain. He remarked that flow of water from an orifice depends not only on the magnitude of the orifice itself, but also on the height of the water in the reservoir; and that a pipe employed to carry off a portion of water from an aqueduct should, as circumstances required, have a position more or less inclined to the original direction of the current. But as he was continued with the law of the velocities of running water as depending upon the depth of the orifice, the want of precision which appears in his results is not surprising (Greenhill 1912).

Seventeenth and Eighteenth Centuries

CASTELLI AND TORRICELLI

Benedetto Castelli and Evangelista Torricelli, two of the disciples of Galileo, applied the discoveries of their master to the science of hydrodynamics. In 1628 Castelli published a small work, *Della misura dell' acque correnti*, in which he suitably explained several phenomena in the motion of fluids in rivers and canals; but he committed a great paralogism in supposing the velocity of the water proportional to the depth of the orifice below the surface of the vessel. Torricelli, observing that in a jet where the water rushed through a small nozzle it rose to nearly the same height with the reservoir from which it was supplied, imagined that it ought to move with the same velocity as if it had fallen through that height by the force of gravity and, hence, he deduced the proposition that the velocities of liquids are as the square root of the head, apart from the resistance of the air and the friction of the orifice. This theorem was published in 1643, at the end of his treatise *De motu gravium projectorum* and it was confirmed by the experiments of Raffaello Magiotti on the quantities of water discharged from different ajutages under different pressures (Greenhill 1912).

BLAISE PASCAL

In the hands of Blaise Pascal hydrostatics assumed the dignity of a science and in a treatise on the equilibrium of liquids, found among his manuscripts after his death and published in 1663, the laws of the equilibrium of liquids were demonstrated in the most simple manner, and amply confirmed by experiments (Greenhill 1912).

STUDIES BY ISAAC NEWTON

Friction and Viscosity

The effects of friction and viscosity in diminishing the velocity of running water were noticed in the *Principia* of Isaac Newton, who threw much light upon several branches of hydromechanics. At a time when the Cartesian system of vortices universally prevailed, he found it necessary to investigate that hypothesis and in the course of his investigations he showed that the velocity of any stratum of the vortex is an arithmetical mean between the velocities of the strata which enclose it; and from this evidently follows that the velocity of a filament of water moving in a pipe is an arithmetical mean between the velocities of the filaments which surround it. Taking advantage of these results, Italian-born French engineer Henri Pitot afterwards showed that the retardations arising from friction are inversely as the diameters of the pipes in which the fluid moves (Greenhill 1912).

Orifices

The attention of Newton was also directed to the discharge of water from orifices in the bottom of vessels.

Waves

Newton was also the first to investigate the difficult subject of the motion of waves.

DANIEL BERNOULLI

Daniel Bernoulli's work on hydrodynamics demonstrated that the pressure in a fluid decreases as the velocity of fluid flow increases. He also formulated Bernoulli's law and made the first statement of the kinetic theory of gases. In fluid dynamics, Bernoulli's principle states that an increase in the speed of a fluid occurs simultaneously with a decrease in pressure or a decrease in the fluid's potential energy. The principle is named after Daniel Bernoulli who published it in his book *Hydrodynamica* in 1738 (Greenhill 1912).

JEAN LE ROND D'ALEMBERT

In fluid dynamics, d'Alembert's paradox (or the hydrodynamic paradox) is a contradiction reached in 1752 by French mathematician Jean le Rond d'Alembert. He proved that for incompressible and inviscid potential flow – the drug force is zero on a body moving with constant velocity relative to the fluid.

LEONHARD EULER

The resolution of the questions concerning the motion of fluids was effected by means of Leonhard Euler's partial differential coefficients. This calculus was first applied to the motion of water by d'Alembert and enabled both him and Euler to represent the theory of fluids in formulae restricted by no particular hypothesis (Greenhill 1912).

GOTTHILF HAGEN

Hagen–Poiseuille equation: In 1839, Hagen undertook careful experiment in brass tubes that enabled him to discover the relationship between the pressure drop and the tube diameter under conditions of laminar flow of homogeneous viscous liquids.

Nineteenth Century

HERMANN VON HELMHOLTZ

In 1858, Hermann Von Helmholtz published his seminal paper "Uber Integrale der Hydrodynamischen Gleichungen, Welche den Wirbelbewegungen entsprechen", in *Journal fur die reine und angewandte mathematk*. So important was the paper that a few years later P.G. Tait published an English translation, "On Integrals of the Hydrodynamical Equations which Express Vortex Motion", in *Philosophical Magazine* (1867). In his paper Helmholtz established his three "laws of vortex motion" in much the same way one finds them in any advanced textbook of fluid mechanics today. This work established the significance of vorticity to fluid mechanics and science in general. For the next century or so, vortex dynamics matured as a subfield of fluid mechanics, always commanding at least a major chapter in treatises on the subject. Thus, H. Lamb's well-known *Hydrodynamics* (1932) devotes full chapter to vorticity and vortex dynamics as does G.K. Batchelor's *An Introduction to Fluid Dynamics* (1967). In due course entire treatises were developed to vortex motion. H. Poincare's *Theorie des Tourbillons* (1893), H. Villat's *Lecons sur la Theorie des Tourbillons* (1930), C. Truesdell's *The Kinematics of Vorticity* (1954), and P.G. Staffman's *Vortex Dynamics* (1992) may be mentioned. Earlier individual sessions at scientific conferences were devoted to vortices, vortex motion, vortex dynamics and vortex flows. Later, entire meetings were devoted to the subject.

The range of applicability of Helmholtz's work grew to encompass atmospheric and oceanographic flows, to all branches of engineering and applied science and, ultimately, to superfluids (today including Bose–Einstein condensates). In modern fluid mechanics, the role of vortex dynamics in explaining flow phenomena is firmly established. Well-known vortices have acquired names and are regularly depicted in the popular media: hurricanes, tornadoes, waterspouts, aircraft trailing vortices (e.g. Wingtip vortices), drainhole vortices (including the bathtub vortex), smoke rings, underwater bubble air rings, cavitation vortices behind ship propellers and so on. In the technical literature, a number of vortices that arise under special conditions also have names: the Karman Vortex Street wake behind a bluff body, Taylor Vortices between rotating cylinders, Gortler Vortices in flow along a curved wall, etc.

JEAN NICOLAS PIERRE HACHETTE

J.N.P. Hachette in 1816-17 published memoirs containing the results of experiments on the spouting of fluids and the discharge of vessels. His object was to measure the contracted part of a fluid vein, to examine the phenomena attendant on additional tubes, and to investigate the form of the fluid vein and the results obtained when different forms of orifices are employed.

Twentieth Century

DEVELOPMENTS IN VORTEX DYNAMICS

Vortex dynamics is a vibrant subfield of fluid dynamics, commanding attention at major scientific conferences and precipitating workshops and symposia that focus fully on the subject.

Vortex atom theory is the new dimension in the history of vortex dynamics, which was done by William Thomson; later it was developed by Lord Kelvin. His basic idea was that atoms were to be represented as vortex motions in the ether. This theory predated the quantum theory by several decades and because of the scientific standing, its originator received considerable attention. Many profound insights into vortex dynamics were generated during the pursuit of this theory. Other interesting corollaries were the first counting of simple knots by P.G. Tait, today considered a pioneering effort in graph theory, topology, and knot theory. Ultimately, Kelvin's vortex atom was seen to be wrong-headed but the many results in vortex dynamics that it precipitated have stood the test of time. Kelvin himself originated the notion of circulation and proved that in an inviscid fluid circulation around a material, contour would be conserved. This result singled out by Einstein in "Zum hundertjahrigen Gedenktag von Lord Kelvins Geburt, Naturwissensschaften"(1924) (title translation: "On the 100th Anniversary of Lord Kelvin's Birth"), as one of the most significant results of Kelvin's work provided an early link between fluid dynamics and topology.

The history of vortex dynamics seems particularly rich in discoveries and rediscoveries of important results, because results obtained were entirely forgotten after their discovery and then were rediscovered decades later. Thus, the integrability of the problem of three-point vortices on the plane was solved in the 1877 thesis of a young Swiss applied mathematician named Walter Grobli. In spite of having been written in Gottingen in the general circle of scientists surrounding Helmholtz and Kirchhoff, and in spite of having been mentioned in Kirchhoff's well-known lectures on theoretical physics and in other major texts such as Lamb's *Hydrodynamics*, this solution was largely forgotten. In an article appeared in the year 1949, it was noted that mathematician J.L. Synge created a brief revival, but Synge's paper was in turn forgotten. A quarter century later a 1975 paper by E.A. Novikov and a 1979 paper by H. Aref on chaotic advection finally brought this important earlier work to light. The subsequent elucidation of chaos in the four-vortex problem, and in the advection of a passive particle by three vortices, made Grobli's work part of "modern science".

Another example of this kind is the so-called "Localized Induction Approximation" (LIA) for three-dimensional vortex filament motion, which gained favour in the mid-1960s through the works of R.J. Arms, Francis R. Hama, Robert Betchov and others, but turns out to date from the early years of the twentieth century in the work of Da Rios, a gifted student of the noted Italian mathematician T. Levi-Civita. Da Rios published his results in several forms but they were never assimilated into the fluid mechanics literature of his time. In 1972 H. Hasimoto used Da Rios' "Intrinsic Equations" (later rediscovered independently by R. Betchov) to show how the motion of a vortex filament under LIA could be related to the non-linear Schrodinger equation. This immediately made the problem part of "modern science" since it was then realized that vortex filaments can support solitary twist waves of large amplitude.

References

Anderson, Jr., J.D., 1997, *A History of Aerodynamics*, Cambridge University Press.

Anderson, Jr. J.D., 1998, "Some Reflections on the History of Fluid Dynamics", in *The Handbook of Fluid Dynamics*, ed. R.W. Johnson, Florida, MIA: CRC Press.

Bachelor, G.K. 1967, *An Introduction to Fluid Dynamics*, Cambridge: Cambridge University Press.

Carroll, Bradley W., 2007, "Archimedes' Principle", Weber State University. Retrieved 2007-07-23.

Garbrecht, G. (ed.), 1987, "Hydrologic and Hydraulic Concepts in Antiquity" in *Hydraulics and Hydraulic Research: A Historical Review,* pp. 1-22, Rotterdam: A.A. Balkema.

Greenhill, A.G., 1912, *The Dynamics of mechanical Flight*, London: Constable.

Lamb, H., 1932, *Hydrodynamics*, Cambridge: Cambridge University Press.

Marshall Clagett, 1961, *The Science of Mechanics in the Middle Ages*, p. 64, Madison, WI: University of Wisconsin Press.

Poincare, H., 1893, *Théorie des Tourbillons*, Paris: Gauthier-Villars.

Staffman, P.G., 1992, *Vortex Dynamics*, Cambridge: Cambridge University Press.

Truesdell, C., 1954, *The Kinematics of Vorticity*, Bloomington, IN: Indiana University Press.

33

Role of Wiener Index in Chemical Graph Theory

A. Dhanalakshmi
K. Srinivasa Rao

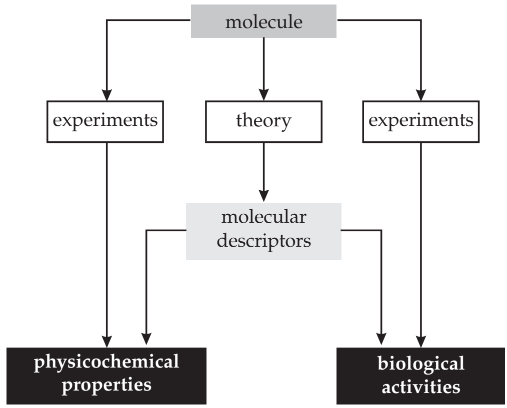
Abstract: We have reviewed the introduction of the Hosoya polynomial and Wiener index. We also reviewed its development and applications in various journals. Here we discuss about the history of the Wiener index, related indices and some of the methodologies used in it so far.

Keywords: Wiener index, Hosoya polynomial, chemical graph theory.

Introduction

IN EARLIER days, Wiener index played a vital role in chemical graph theory. Application of topological indices in biology and chemistry began in 1947. The Chemist Harold Wiener (1947) introduced the Wiener index to demonstrate correlations between physicochemical properties of organic compounds and the index of their molecular graphs.

Molecular descriptors are numerical values obtained by the quantification of various structural and physicochemical characteristics of the molecule. It is envisaged that molecular descriptors quantify these attributes so as to determine the



Scheme of Molecular Descriptor

behaviour of the molecule and the way the molecule interacts with a physiological system. Since the exact mechanism of drug activity is unknown in many cases, it is desirable to start with descriptors spanning as many attributes of the molecules as possible and then assess their ability to predict the desired activity/property.

Topological indices of a simple graph are numerical descriptors that are derived from the graph of chemical compounds. Such indices based on the distances in graph are widely used for establishing relationships between the structure of molecular graphs and nanotubes and their physicochemical properties.

Wiener (1947) originally defined his index on trees and studied its use for correlations of physicochemical properties of alkanes, alcohols, amines and their analogous compounds as:

$$WI = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d(u,v),$$

where $d(u, v)$ denotes the distance between vertices u and v.

The Hosoya polynomial of a graph is a generating function about distance distributing, introduced by Hosoya in 1988 and for a connected graph G is defined as (Babujee et al. 2012):

$$H(x) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} x^{d(u,v)}$$

In a series of papers, the Wiener index and the Hosoya polynomial of some molecular graphs and nanotubes are computed. For more details about the Wiener.

Ivan Gutman et al., introduced the system of molecular descriptor and its applications in QSPR/QSAR (Quantitative structure property/activity relationships). Ivan Gutman and Oskar E. Polansky (1986) suggested the conversion of the structure of a molecule into a graph and introduced the concept of graph energy, topological indices. Babujee and Sengabamalar (2012) explained how Wiener index correlates with properties of organic compounds and found Wiener index of some common cycles, paths, complete graph and star graph and so on. K. Tilakam et al. (2014) obtained the Wiener index of some graphs using Matlab.

Mohamed Essal et al. (2011) derived some theoretical results for the Wiener index, degree distance and the hyper Wiener index of a graph. Sandi Klavžar and Ivan Gutman (1996) compared the Schultz molecular index with the Wiener index.

Sandi Klavžar (2008) presented the applications of chemical graph theory and used cut method to find the topological indices: Wiener index, Szeged index, hyper-Wiener index, the PI index, weighted Wiener index, Wiener-type indices, and classes of chemical graphs such as trees, benzenoid graphs and phenylenes.

Wiener (1947) instructed to compute in a simple way to find the path number. Multiply the number of carbon bonds on one side of any bond by those on the other side. W is the sum of these values for all bonds. Let T be a tree with N vertices and e one of its edges (bonds). Let also $N_1(e)$ and $N_2(e) = N - Nj(e)$ be the numbers of vertices of the two parts of $T - e$.

$$W = \sum_{e} N_1(e) N_2(e)$$

where the summation is over all N_1 edges of T.

Sonja Nikolić et al. (1995) reviewed the definitions and methods of computing the Wiener index. They pointed out that the Wiener index is a useful topological index in the structure–property relationship because it is a measure of the compactness of a molecule in terms of its structural characteristics, such as branching and cyclicity. Also, they did a comparative study between the Wiener index and several of the commonly used topological indices in the structure–boiling point relationship. Developments such as an extension of the Wiener index into its three-dimensional version are also mentioned.

Conclusion

Reviewing the mathematical properties and the chemical applications of the Wiener index, it is one of the best understood and most frequently used molecular descriptors. It has numerous applications in the modelling of physicochemical, pharmacological and biological properties of organic molecules.

References

Babujee, J. Basker and J. Sembagamalar, 2012, "Tropological Indices and New Graph Structures", *Applied Mathematical Sciences* (AMS), 6(88): 4387-95.

Essalih, Mohamed, Mohamed el Marraki and Gabr el Hagri, 2011, "Calculation of Some Topological Indices of Graphs", *Journal of Theoretical and Applied Information Technologies*, 3(2): 122-27.

Gutman, Ivan and Oskar E. Polansky, 1986, *Mathematical Concepts in Organic Chemistry*, Berlin/Heidelberg: Springer-Verlag.

Nikolić, Sonja and N. Trinajstic, 1995, "The Wiener Index: Development and Application", *Croatica Chemica Acta* (CCA) 68(1): 105-29.

Klavžar Sandi, 2008, "A Bird's-Eye View of the Cut Method and a Survey of Its Applications in Chemical Graph Theory", *MATCH Commun. Math. Comput. Chem.*, 60**:** 255-74.

Klavžar Sandi and Ivan Gutman, 1996, "A Comparison of the Schultz Molecular Topological Index with the Wiener Index", *J. Chem. Inf. Comput. Sci.*, 36: 1001-03.

Thilakam, K. and A. Sumathi, 2014,"Wiener Index of a Cycle in the Context of Some Graph Operations", *Annals of Pure & Applied Mathematics* (APAM), 5(2): 183-91.

Wiener, H., 1947, "Structural Determination of Paraffin Boiling Points", *J. Amer. Chem. Soc.* 69: 17-20.

34

The Origin of Semiring-valued Graph

Ramya
T.N. Kavitha

Abstract: We discuss the origin of *S*-valued graph and its application fields. The semiring-valued graph is defined as the combination of graph and algebraic structure. Various types of *S*-valued graphs are defined by many persons. From those discussions here we talk about a few of them, for example, vertex domination on *S*-valued graph, degree regularity on edges of *S*-valued graph, homomorphism on *S*-valued graph, and vertex domination number in *S*-valued graph; using these discussions we try to find a new type of *S*-valued graph in future.

Introduction

THE origin of *S*-valued graph was in 1934. H.S. Vandiver introduced the semiring and studied "the algebraic structure of ideals in rings". Further, Jonathan Golan introduced the notion of *S*-valued graphs, i.e. *S*-semiring. In the year 2015, M. Chandramouleeswaran introduced the semiring-valued graph in the *International Journal of Pure and Applied Mathematics*. In 2016, S. Jeyalakshmi introduced vertex domination on *S*-valued graph in the *International Journal of Innovative Research in Science*. It was followed by the authors S. Mangala Lavanya and S. Kiruthiga Deepa and they introduced "degree regularity on edges of *S*-valued graph" in the same year.

Further, M. Rajkumar (2016) introduced "the homomorphism on *S*-valued graph". In 2016, S. Jeyalakshmi presented the paper "Strong and Weak Vertex Domination on *S*-valued Graph" in *The International Journal of Pure and Applied Mathematics*. The author motivated the notion of *S*-valued graphs. "K-colourable *S*-valued graph" was introduced by T.V.G. Shriprakash as he published in the *International journal of Pure and Applied Mathematics* (2017). In July 2017, the *S*-valued definition was introduced by S. Jeyalakshmi in the *Mathematical Science International Research Journal*. She introduced the definition the vertex $v \in Gs$ is said to be a weight dominating vertex if $\sigma(u) \geq \sigma(v)$, for all $u \in V$. M. Sundar introduced the applied graph theory paper in the year 2017. He states that the products of graph have lead several areas of research in graph theory. Algebraic graph theory can be viewed as an extension of graph theory in which algebraic methods are applied to problems about graphs.

Origin of S-valued Graph

In 1934, H.S. Vandiver introduced the semiring in the study of algebraic structure of ideals in rings. Further, Jonathan Golan introduced the notion of "S-valued graphs". That is also known as semiring-valued graphs. There are some major applications of it in such fields as social sciences, communications, networks and algorithms designs.

Semiring-valued Graph

M. Chandramouleeswaran introduced the semiring-valued graph in the *International Journal of Pure and Applied Mathematics* in 2015. He mainly combined the algebraic structure with the graph that is known as semiring-valued graph. It has the other notation as *S*-valued graph.

He defined a semiring $(S, +, *)$ as an algebraic system with a non-empty set S together with two binary operations + and * such that

1. $(S, +, *)$ is a monoid.
2. $(S, *)$ is a semigroup.

3. For all $a, b, c \in S$, $a \times (b + c) = a \times b + a \times c$ and $(a + b) \times c = a \times c + b \times c$.
4. $0 \cdot x = x \cdot 0 = 0 \ \forall \ x \in S$.

This can be applied to find certain social network problems.

Vertex Domination on S-valued Graph

S. Jeyalakshmi published an article "Vertex Domination on *S*-valued Graph" in the *International Journal of Innovative Research in Science* (September–October 2016).

She defined a set $D \subseteq V$ as a dominating vertex set of G, if $\forall\ v \in V - D, N(v) \cap D \neq \varphi$. A dominating set D is a minimal dominating vertex set if no proper subset of D is a vertex dominating set in G.

The study of domination is the fastest growing area in graph theory. For that she introduced the notion of vertex domination on *S*-valued graphs and proof of some simple properties.

Degree Regularity on Edges of S-valued Graph

In the *Journal of Mathematics*, S. Mangala Lavanya and S. Kiruthiga Deepa published an article entitled "Degree Regularity on Edges of *S*-valued Graph".

They defined a domination set S as a minimal edge dominating set if no proper subset of S is an edge dominating set in G. A *S*-valued graph G^s s is said to be d_s- edge regular if for any $e \in E$, $\deg_s (e) = (|N_s(e)|_s, |N_s(e)|)$.

The authors studied the regularity conditions on the *S*-valued graph. Further, the moved about the study of the edge-degree regularity of the *S*-valued graph. Then they discussed about the edge-degree regularity of *S*-valued graphs in thier paper.

Homomorphism on S-valued Graph

M. Rajkumar (2016) introduced the homomorphism on *S*-valued graph. He published in the *International Journal of Engineering and Technology*. He derived the concepts of homomorphism and isomorphism between two *S*-valued graphs.

According to him, let S_1 and S_2 be semirings. A function $\beta : S_1$

$\rightarrow S_2$ is a homomorphism of semirings if $\beta\,(a + b) = \beta\,(a) + \beta\,(b)$ and $\beta\,(a \cdot b) = \beta\,(a) \cdot \beta\,(b)$ for all $a, b \in S_1$.

The author has introduced the notion of homomorphism and isomorphism on S-valued graphs. We study whether the isomorphism of graphs prevents the regularity conditions or not.

Further, M. Rajkumar and M. Chandramouleeswaran are going to extend S-valued homomorphism into S-valued isomorphism.

Strong and Weak Vertex Domination on S-valued Graph

S. Jeyalakshmi defined a dominating set X is said to be a strong dominating set if for every vertex $u \in V - X$ then is a vertex $v \in X$ with $\deg(v) \geq \deg(u)$ and we conclude that the vertex u is adjacent to v.

A dominating set X is said to be a weak dominating set if for every vertex $u \in V - X$ there is a vertex $v \in X$ with $\deg(v) \leq \deg(u)$ and u is adjacent to v.

The study of domination in graph theory is the fastest growing area. So she introduced the notion of strong and weak vertex domination on S-valued graphs and proof of some simple results.

Vertex Domination Number in S-valued Graph

In 2016 S. Jeyalakshmi presented the vertex domination number in S-valued graph in the *International Journal of Innovative Research in Science, Engineering and Technology*.

Consider the S-valued graph $G^S = (V, E, \sigma, \psi)$. Let $u \in V$ be a vertex of G^S whose degree in the crisp graph G of G^S is equal to (G). That is $(G) = \deg(u)$. Let $w \in V$ be a vertex of G^S whose degree in the crisp graph G *of* G^S equal to $\Delta(G)$. That is $\Delta(G) = \deg(w)$. The minimum degree and the maximum degree of the S-valued graph G^S are defined as

$$\underset{u \in V}{Min} \deg_s(u) = \left(\sum\nolimits_{v \in N_s(u)} \psi(uv), \delta(G)\right) = \delta_s(G^S)$$

and

$$\underset{w \in V}{Max} \deg_s(w) = \left(\sum\nolimits_{v \in N_s(w)} \psi(wv), \Delta(G)\right) = \Delta_s(G^S).$$

We analyse the vertex domination number in *S*-valued graph. The authors Jeyalakshmi. S and Chandramouleeswaran. M gave some results on the bounds of the weight-dominating vertex number of *S*-valued graphs.

K-colourable *S*-valued Graph

K-colourable *S*-valued graph was introduced by T.V.G. Shriprakash in the year 2017 (April). He published in the *International Journal of Pure and Applied Mathematics*.

An *S*-valued graph G^S is said to be k-colourable, if it has a proper vertex regular or total proper colouring such that $|C| = k$.

In proper, the vertex colouring of the graph *G*, the vertices that receive the common colour are independent. The vertices that receive a particular colour make up a colour class.

In any chromatic partition of $V(G)$, the parts of the partition constitute the colour classes, which allow an equivalent way of defining the chromatic number.

Finally, the author worked about the upper bounds of *K*-colourable *S*-valued graphs.

Total Weight Domination Vertex Set on S-valued Graph

In the year 2017 (July), this paper was introduced by S. Jeyalakshmi in the *Mathematical Science International Research Journal*. She says a vertex v in G^s said to be a weight-dominating vertex if $\sigma(u) \leq \sigma(v)$, for all $u \in V$.

A subset $D \subseteq V$ is said to be a weight-dominating vertex u set of G^S if for each $v \in D$ $\sigma(u) \leq \sigma(v)$, for all $u \in N_s(v)$. If Ns $(T_D{}^S = V_s$, then $T_D{}^S$ is called a total weight-dominating vertex set of G^s.

Berge introduced the domination in graphs. Nowadays the most leading area is vertex domination. The author moved and worked about the domination of vertex set on *S*-valued graph. So he introduced "the total weight domination vertex set on *S*-valued graphs". They give some properties and simple proofs.

Cartesian Product of Two S-valued Graph

This paper was introduced by M. Sundar in 2017. Products of graph have lead several areas of research in graph theory. Algebraic graph theory can be viewed as an extension of graph theory in which algebraic methods are applied to problems about graphs.

He defined

Let

$G_1^S = (V_1, E_1, s_1, \psi_1)$

where

$V_1 = \{v_i \mid 1 \leq p_1 \leq p_1\}$,

$E_1 \subset V_1 \times V_1$

and

$G_2^s = (V_2, E_2, s_2, \psi_2)$

where

$V_2 = \{v_2 \mid 1 \leq j \leq p_2\}$,

$E_2 \subset V_2 \times V_2$

be two given S-valued graphs.

$V_1 \times V_2 = \{w_{ij} = (v_{i.}\ u_j) \mid 1 \leq i \leq p_1, 1 \leq j \leq p_2\}$; $E_1 \times E_2 \subset V_1 \times V_2$.

The Cartesian product of two S-valued graphs G_1^S and G_2^S = is a graph defined as

$G^S = G_1^S G_2^S = (V = V_1 \times V_2, E = E_1 \times E_2, \sigma = \sigma_1 \times \sigma_2, \psi = \psi_1 \times \psi_2)$,

where

$V = \{w_{ij}(v_i, u_j)\ v_i \in V_1$ and $u_j \in \mathrm{V}_2$

and two vertices w_{ij} and w_{kl} are adjacent if $i = k$ and $u_j u_l \in E_2$ or $j = l$ and $v_i v_k \in E_1$.

In this paper, the author discussed the concept of Cartesian products of two S-valued graphs.

Neighbourly Irregular S-valued Graphs

M. Rajkumar (2017) introduced neighbourly irregular S-valued in the *International Journal of Pure and Applied Mathematics*.

A graph is said to be regular if every vertex has equal degree, otherwise it is called a irregular graph.

A graph in which for each vertex v of G, the neighbours of v have distinct degrees, is called a locally irregular graph.

A connected graph is said to be highly irregular if for every vertex v, u, w $N(v)$, $u \neq w$ implies that deg $u \neq$ deg w. That is, every vertex is adjacent only to vertices with distinct degrees.

M. Chandramouleeswaran and others introduced the notion of S-valued graphs and regularity on S-valued graphs. Here they introduced the notion of irregularity conditions on S-valued graphs.

First they successfully define the locally r-regular graph which has equal degree of vertices. Further, they define the locally r-irregular graphs which have the distinct number of vertices. It is more generally in a way that the irregularity conditions on a crisp graph.

Conclusion

S-valued graph is the combination of algebraic structure and a graph. Its origin is discussed in this paper. Further we try to develop the S-valued graph in some different way.

References

Jeyalakshmi, S. and M. Chandramouleeswaran, 2017, "Total Weight Dominating Vertex Set on S-Valued Graphs", *Mathematical Science International Research Journal*, 6(2): 147-50.

Rajkumar, M. and M. Chandramouleeswaran, 2016, "Homomorphisms on S-Valued Graphs", *International Journal of Emerging Trends in Science and Technology*, 3(10): 4695-4703.

———, 2017, "Neighbourly Irreregular S-Valued Graphs", *International Journal of Pure and Applied Mathematics*, 112(5): 23-30.

Shriprakash, T.V.G. and M. Chandramouleeswaran, 2017, "K-Colourable S-Valued Graphs", *International Journal of Math. Sci. & Engg. Appls.*, 11(1): 151-57.

35

History of Optimization Models in Evolutionary Algorithms

K. Bharathi

Abstract: The collection of optimization techniques, which is functioning based on metaphors of biological processes, is termed evolutionary algorithm. A multi-objective optimization problem has several incompatible objectives with a set of Pareto optimal solutions. A developed set of solutions as population, evolutionary algorithms in multi-objective optimization is able to estimate the Pareto optimal position. We have a review and overview of development in evolutionary algorithm for multi-objective optimization during the last sixteen years. Here, we discuss about the history of the framework, related algorithms development and their applications and some of the methodologies used in it so far.

Keywords: Algorithms, evolutionary algorithm, multi-objective optimization problem, Pareto evolutionary algorithm, new quantum evolutionary algorithm.

Introduction

EVOLUTIONARY algorithms (EAs) are the machine learning approaches from natural collection in the biological world (Sharma et al. 2017). EAs vary from more established optimization

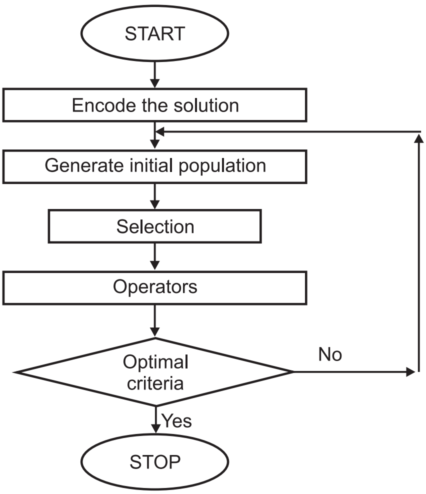


fig. 35.1: Basic evolutionary algorithm

techniques, where EAs involve a set of solutions called population. The iterations of an EA involve an aggressive selection that includes feasible solutions. A set of solutions is operated by using one or more operations to get a best optimal solution. If we have more than one criterion to be optimized with several conditions, said to be a constraint equation, such a problem is named as multi-objective optimization (Nanvala 2011). The processing method of the EA is presented in *fig*. 35.1.

Representation of Evolutionary Algorithms

The relation made to the solution space elements by encoding the phenotype values to a genotype value is said to be the relation of representation. Since evolutionary algorithm can make use of genotype representation as an encoded solution so as to precede in the algorithm it is represented in many ways. The different ways of evolutionary representation is genetic algorithm, genetic programming, evolution strategy and many more.

GENETIC ALGORITHM

The development of genetic algorithms was handled by many authors to extend the solution space of the model of optimality and

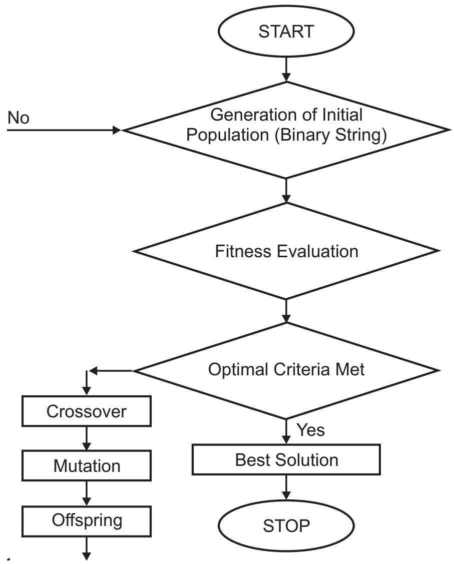


fig. 35.2: Flow of genetic algorithm

to find the optimal result for the technical real world formulation. The improved genetic algorithms are stochastic in natural world models with thier condition of sufficiency is applied to technical real world formulation in *fig*. 35.2.

GENETIC PROGRAMMING

A unique representation in the model of genetic algorithm is referred to as genetic programming. The real value solution of the phenotype is encoded as a graphical tree-shaped genome which is the vital position of genetic programming. All the sets of solution to the problem will have a characterized representation as a tree-shaped genotype solution with its fitness condition applied to each of the solutions. The generalized approach of genetic programming is scheduled in *fig*. 35.3.

EVOLUTION STRATEGY

Evolution strategy, which works out from the year 1968, is an old method evolved before genetic algorithm. This strategy gives the best result to the models involving continuous variable

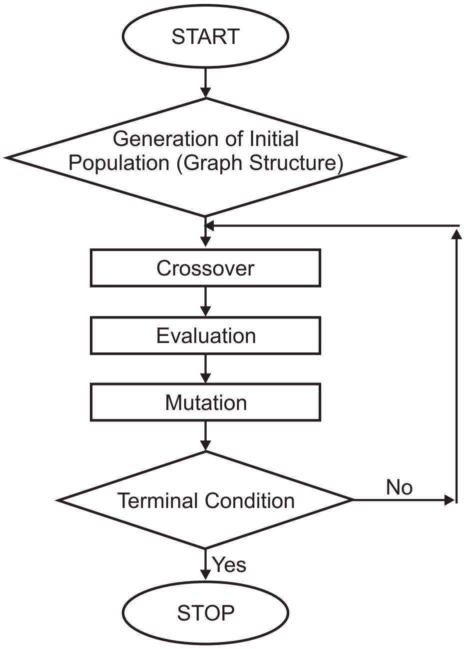


fig. 35.3: Flow of genetic programming

rather than the discrete variable model. Strategy of evolutionary encoding system differs from its genome as a real value vector space applied in machine-based model. The encoded structure differs from nature-based operating tool comparing with other representations. In general, the operators are differed by their names such as endogen instead of crossover and size of the step is replaced by the value of mutation. Especially, the gene of the bits of the representation is a real valued parameter-setting allocation. Hence, the generalized flow of the evolutionary strategy is shown in *fig*. 35.4.

EVOLUTION PROGRAMMING

Evolution programming, works out from the year 1960, was introduced by Lawrence. This programming method gives the best result to the models involving continuous variable rather than the discrete variable model and its similarity to evolution strategies. Evolution programming in the encoding system differs

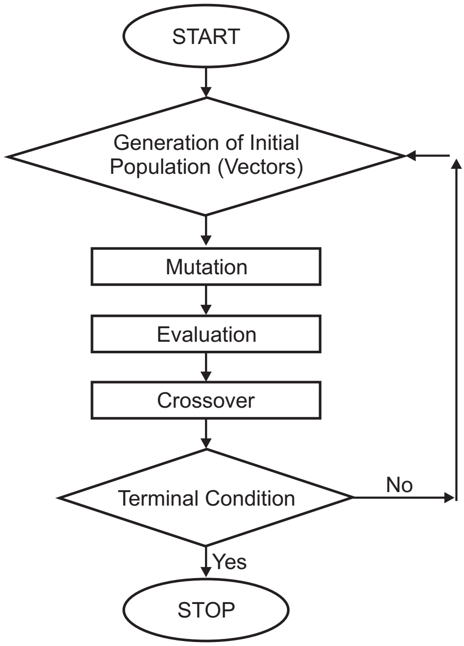


fig. 35.4: Flow of the evolution strategy

to its genome as a real-value vector space applied in machine-based model. The encoded structure differs from nature-based operating tool comparing with other representations. In general, the operators were differed by their names such as size of the step is replaced by the value of mutation. Especially, the gene of the bits of the representation is as a real-valued parameter setting allocation. Hence, the generalized flow of the evolution programming is shown in *fig*. 35.5.

Multi-objective Optimization

A multi-objective optimization problem involves a number of objective functions which are to be either minimized or maximized. As in a single-objective optimization problem, the multi-objective optimization problem may contain a number of constraints which any feasible solution (including all optimal solutions) must satisfy. Since objectives can be either minimized or maximized, we state the multi-objective optimization problem in its general form:

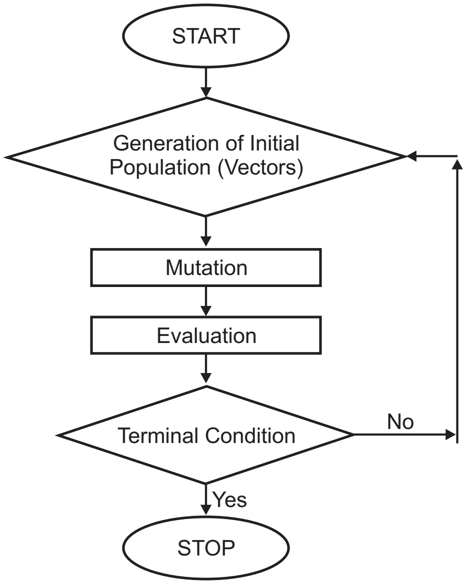


fig. 35.5: Flow of evolution programming

Optimize	$f_m(x)$	$m = 1, 2, 3, ..., M$
Subject to	$g_n(x) \geq 0$	$n = 1, 2, 3, ..., N$
	$h_K(x) = 0$	$K = 1, 2, 3, ..., K$
	$x_i^L \leq x_i \leq x_i^{U}$	$i = 1, 2, 3, ..., n$
	$\forall\ x \geq 0,$	

where M is the number of objective functions, N is the number of unequal constraint, K is the number of equality constraint, L denotes the lower limit value and U denotes the upper limit value.

The main difference between the single-objective and multi-objective optimization is that in the multi-objective optimization, the objective functions constitute a multi-dimensional space, in addition to the usual decision variable space. This additional M-dimensional space is called the objective space, $Z \subset R^M$. For each solution, x in the decision variable space, there exists a point $z \in R^M$ in the objective space.

Conclusion

A multi-objective optimization problem has many complicated objectives with a set of Pareto optimal solutions. A developed set of solutions as population, evolutionary algorithms in multi-objective optimization estimates the Pareto optimal position. Here, we have dealt with a general overview of evolutionary algorithms to multi-objective optimizations in the past sixteen years. We have discussed the algorithms, methodology used, applied field and significant works. Also the most delegate existing study trends were discussed and provided the advantages present in using EAs in the different fields.

References

Hu, Guanzhong, Shiyou Yang, Yuling Li and Shafi Ullah Khan, 2016, "A Hybridized Vector Optimal Algorithm for Multi-Objective Optimal Designs of Electromagnetic Devices", *IEEE Transactions on Magnetics*, March, 52(3).

Kiraly, Andras and Janos Abonyi, 2016, "Optimization of Multiple Traveling Salesmen Problem by a Novel Representation Based Genetic Algorithm", *10th International Symposium of Hungarian Researchers on Computational Intelligence and Informatics*, pp. 82-91.

Nanvala, H., 2011, "Use of Genetic Algorithm Based Approaches in Scheduling of FMS: A Review", *International Journal of Engineering Science and Technology* (*IJEST*), 3(3): 1936-42.

Sharma, Arun Kumar, Rituparna Datta, Maha Elarbi, Bishakh Bhattacharya and Slim Bechikh, 2017, "Practical Applications in Constrained Evolutionary Multi-objective Optimization", in *Recent Advances in Evolutionary Multi-objective Optimization*, ed. Slim Bechikh, Rituparna Dutta and Abhishek Gupta, pp. 159-79, Switzerland: Springer.

Shu, Meng and Dai Bo, 2012, "A Novel Method for Solving the Multiple Traveling Salesmen Problem with Multiple Depots", *Chin. Sci. Bull.*, 57: 1886-92.

Zhang, H. and R. Chen, 2008, "Research on Coarse-grained Parallel Genetic Algorithm-based Grid Job Scheduling", *Proceedings of the*

Fourth International Conference on Semantics, Knowledge and Grid, pp. 505-06.

Zhu, Zhaomeng, Gongxuan Zhang, Miqing Li, Xiaohui Liu, 2016, "Evolutionary Multi-Objective Workflow Scheduling in Cloud", *IEEE Transactions on Evolutionary Computation*.

Zupan, H ., N. Herakovic and J. Zerovnik, 2015, "A Hybrid Metaheuristic for Job-shop Scheduling with Machine and Sequence-dependent Setup Times", *Proceedings of the International Symposium on Operational Research in Slovenia* (*SOR*), pp. 129-34.

36

Graph Theory for Detection of Crime

C. Yamuna
T.N. Kavitha

Abstract: Nowadays, we are facing many problems, related to crime and to solve these problems mathematics is being used. In this paper we shall examine the role of one branch of mathematics i.e. graph theory, in addressing such problems of society. We have chosen to present mathematic related topic from the field of graph theory because graphs have wide-ranging applicability and it is possible to bring a previously unfamiliar scientist to the frontiers of research rather quickly. The graph theory has been used beyond simple problem formulation. Sometimes, a part of a large problem corresponds exactly to a graph-theoretic problem, and that problem can be completely solved.

Keywords: Accusation, crime, logical, suspect, node, graph.

Introduction

HOW to solve the crime using the graph theory? We have chosen three persons named Alice, Bob and Charlie or simply we call *A*, *B* and *C*. Each of them has given a statement, regarding an accusation. We can form a graph with three nodes and solve who has done the crime out of those three persons.

Can it be possible to work out the three cases? The answer is yes, and it is not so difficult. Usually if a problem is formulated through graph theory, it can be solved by the process of simplifications, consideration of important aspects such as changing of relationship frequently or looking into strenghth of effects, and omission of unimportant aspects.

If someone suggests that the graph theory is a panacea and by itself we can solve a large number of problems, we can disclaim that statement. But graph theory is just one tool, which sometimes solves problems and sometimes gives us insights. It usually has to be used along with many other tools, mathematical or otherwise. Hopefully, the use of the graph theory can help us to understand in small ways the large problems that confront our society, and some possible solutions. Finally, let us remember that applied mathematics develops when it is in close contact with applications.

Resolving of a Crime Using Graph

We have chosen three persons named Alice, Bob and Charlie or simply we call *A*, *B* and *C*. Each of them has made a statement, containing an accusation. *A* says I am not the thief. *B* says *A* is the thief and *C* says I am not the thief, here we also know that only one person is telling the truth.

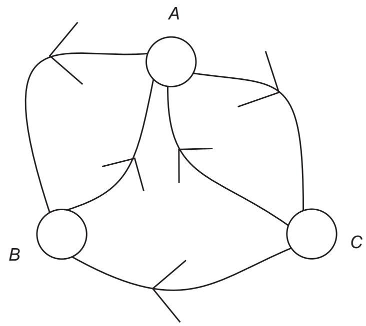
Suppose *A* was the thief, then *B* and *C* would be telling the truth, but only two persons appear to be telling the truth when we know that there should only one who is the thief. If *B* is the thief then both *A* and *C* are telling the truth. That way we have to eliminate by using the graph in order to come to the conclusion.

But what if you have more suspects say four or five and with a more complicated set of accusations, can we find a quicker method to solve this kind of problems?

Let us take it. You suspect and represent them as a small dot in space or a node now. *B* accuses *A*, so you can represent this by drawing a direct line from *B* to *A*.

C says I am not the thief, so we think of this as being equivalent to accusing everyone else both *B* and *A*. So *C* has a line going to both *B* and *A*.

Similarly, *A* says I am not the thief. So *A* accuses both *B* and *C*, this kind of diagram is called a graph. So we use it to solve the crime.



Let us go over the cases, we considered if any of the thieves will reconstruct the graph ignoring all the lines coming out of *B* accuses *A*. So they have a line directed into only one person is telling the truth; this means there should only be one line going into *A*, since there are two lines going to *A*, *A* cannot be the thief.

If we consider *B* as the thief again, then ignoring all the lines going out of *B*. We see that, there's still two lines direction to be the two accusations from *A* and *C*. This would mean that the *A* and *C* were telling the truth. So, this cannot be the case. Finally, we come to see *B* accusation from *A*. So it has no line directed into *C*. But there is an accusation from *A*. So there's only one line going into *C* and only one person telling the truth. This is the only logically consistent case. So *C* is the thief.

Algorithm

Step 1: Take a node, add up the number of lines going into.

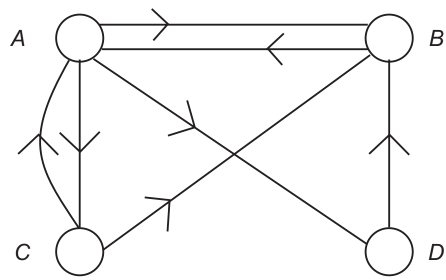
Step 2: Count the number of lines going in (this gives us a number of people telling the truth if that person is a thief, then move on to the next node and repeat so for all three persons).

Step 3: This method is not so short because fo the process of going through the statement of each person and move from case-by-case.

Resolving Crime Using Graph where more than three Persons are Involved

Now, we have chosen four people *A*, *B*, *C* and *D*, and only one of them is telling the truth. *A*, *C* and *D* the give the same statement that say, *B* is a thief.

We can draw a graph filling in the lines coming from *A*, *C*, *D* looking at *B*.



We have constructed two lines going to *B* and from *B*, three lines are going, *C* has one and *D* has two. The list gives us a number of people that will be telling the truth. So, if one person were telling the truth, *C* is again the thief. However, we know that, free people are having the troops then we can immediately see that *B* is the only case of free people, and therefore *C* can be identified as the thief. If there are two persons telling the truth, then we have two possible solutions *A* or *D*. So, it could be either of them.

However, with more information we can tell exactly which one, but we do know that it is definitely not *B* or *C*. So, we form a list that gives us all of the possible solutions for any set of suspects and accusations. We have to form a list and match the numbers to study how many persons are telling the truth.

Applications

This method of solving problems can be used in artificial intelligence and computer science. Mathematicians and scientists need to develop system solutions. That can be easily implemented incorporating into computer program.

Conclusion

One should have computer coding to follow an algorithm like this, which is remarkably easy. So, one can write a program using this method to find all of the possible solutions. For a larger set of the suspects, say 100 of them, it can be done in a short span of time.

References

Gephi.org., 2017, "About", retrieved an 29 October 2017, from https://gephi.org/about/.

IBM, 2017, "IBM i2 Analyst's Notebook", retrieved from https://www.ibm.com/us-en/marketplace/analysts-notebook.

Jacomy, M., T. Venturin, S. Heymann and M. Bastian, 2014, "ForceAtlas2, a Continuous Graph Layout Algorithm for Handy Network Visualization Designed for the Gephi Software", *Plos One*. https://doi.org/10.1371/journal.pone.0098679.

Sarvari, H., E. Abozinadah, A. Mbaziira and D. McCoy, 2014, "Constructing and Analyzing Criminal Networks", *IEEE Security and Privacy Workshops*. Retrieved from http://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=6957290.

37

A Discussion on Real Life Application of Mathematics

T.N. Kavitha
B. Akila

Abstract: The area of study in the history of mathematics is primarily discovered. We investigate the origin of discoveries in mathematics and, to a lesser extent, an investigation into the mathematical methods and notation of the past. The study of mathematics as a study in its own right began in the sixth century BCE with Pythagoras, who coined the term "mathematics" from the ancient Greek term *mathema*, meaning "the subject of instruction". Greek mathematicians greatly refined the methods and expanded the subject matter of mathematics. In this paper, we try to present the application of calculus in the transition curve of a rail track.

Keywords: Calculus, transition, Greek mathematics.

Introduction

FROM time immemorial, mathematics is part and parcel of human life, it most probably began with counting. Here is an attempt to learn the history of mathematics and thus we get to know some of the greatest mathematical minds and their contributions.

Mathematics is the mother of all intentions in this world. The foremost three developments in the civilization of human being are: (1) invention of fire, (2) invention of maths, and (3) invention of the circle. Because, based on these, man invented a number of scientific and medical equipment.

The first invention of man when he started to learn mathematics was a circle or the wheel. With the help of the circle or wheel, he created most scientific equipment and machinery.

There is much application of integral calculus in real life and engineering. Application of the equation of the curve is discussed here.

Cartesian Form of the Equation of the Curve

Let P be any point on a given curve and Q a neighbouring point. Let arc $AP = s$ and arc $PQ = \delta s$. Let the tangents at P and Q make angles Ψ and $\Psi + \delta\Psi$ with the X axis, so that the angle between the tangents at P and $Q = \delta\Psi$. In moving from P to Q through a distance δs, the tangent turns through the angle $\delta\Psi$. This is called the total bending or total curvature of the arc PQ.

The average curvature of arc $PQ = \frac{\delta\Psi}{\delta s}$. The limiting value of average curvature when Q approaches P is defined as the curvature of the curve at p.

$$\text{Thus, curvature } K \text{ (at } P) = \frac{d\Psi}{ds}. \qquad (1)$$

Since $\delta\Psi$ is measured in radians, the unit of curvature is radians per unit length, e.g. radians per centimetre.

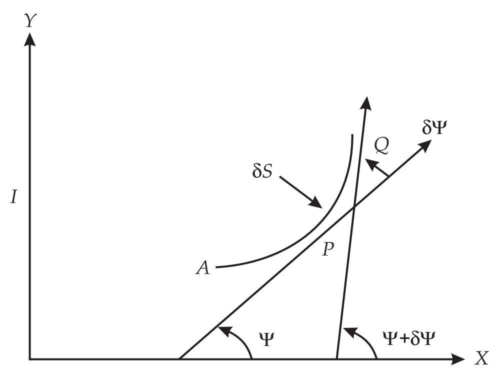


fig. 37.1 Radius of Curvature

Definition

Radius of curvature: The reciprocal of the curvature of a curve at any point P is called the radius of the curvature at P and is denoted by ρ, so that $\rho = \frac{ds}{d\Psi}$.

Radius of Curvature for Cartesian Curve

Let the equation of the curve in Cartesian form be $y = f(x)$, then $\tan \alpha = dy/dx = y$ or $f(x)$ so that $\alpha = \tan^{-1} y_1$ and hence, $d\alpha/dx = 1/(1 + y_1^2)$.

We know

$$ds/dx = \frac{ds}{dx} = \sqrt{1+y_1^2}.$$

The curvature at a point S (x, y) is expressed in terms of the first derivative (y_1) and second derivatives (y_2) of the function $f(x)$ by the formula. Therefore,

$$k = \frac{da}{ds} = \frac{da}{da}\frac{dx}{ds} = \frac{y_2}{1+y_1^2} - \frac{1}{\sqrt{1-y_1^2}}.$$

Thus, we obtain

$$k = \frac{da}{ds} = \frac{y_2}{\left(1+y_1^2\right)^{3/2}}$$

The absolute value of the ratio $\mathrm{k} = \frac{d\Psi}{ds}$ is called the mean curvature of the arc PQ. In the limit as $ds \to 0$, we obtain the curvature of the curve at the point P. $k = \lim_{ds \to 0}\left|\frac{d\Psi}{ds}\right|$. From this definition, it follows that the curvature at a point of a curve characterizes the speed of rotation of the tangent of the curve at the point. The following is the application of this concept in a curve shape turning of the railway track. Here it shows that mathematics has an application in real life.

Transition Curve of a Rail Track

A track transition curve, or spiral easement, is a mathematically calculated curve on a section of highway, or railroad track, in

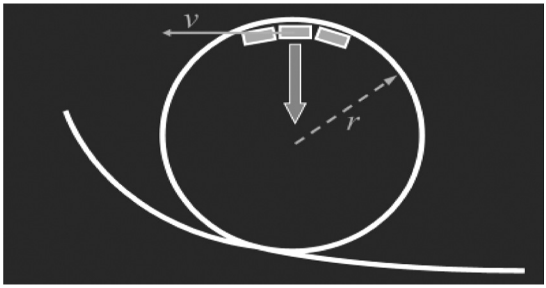


fig. 37.2: Tangent point at intersection of curve

which a straight section changes into a curve. It is designed to prevent sudden changes in lateral (or centripetal) acceleration. In an aerial view, the start of the transition of the horizontal curve is at infinite radius, and at the end of the transition, it has the same radius as the curve itself and so forms a very broad spiral. At the same time, in the vertical plane, outside of the curve is gradually raised until the correct degree of bank is reached.

If such an easement was not applied, the lateral acceleration of a rail vehicle would change abruptly at one point (the tangent point where the straight track meets the curve) with undesirable results. With a road vehicle, the driver naturally applies the steering alteration in a gradual manner, and the curve is designed to permit that, by using the same principle of radius of curvature.

The transition curves in modern highway and railway construction are route elements equally crucial as alignment and curve (circular). In order to prevent a sudden change of the centrifugal force, the transition curve must be applied due to the impact of the motion in a sharp curve. Over the years, the application of the adorned has become widespread in many countries. However, in this study, in order to eliminate the problems concerning the road dynamics, created by adorned for vehicles at high speed, sinusoid their fundamental mathematical expression, calculation of point coordinates, driving, dynamic characteristics of sinusoid are described the curvature and lateral impact along the sinusoid is presented. Sinusoid is dealt with as an ideal curvature diagram which has curve and superelevation ramp in the articles.

At high speeds trains cannot pass abruptly from a straight stretch of track to a circular track of high curvature. In order to make the change of direction gradual, engineers make use of transition curves to connect the straight part of a track with a circular track. Generally, arcs of cubical parabola are employed.

Suppose the transition curve on a railway track has the shape of an arc of the cubical parabola $y = (1/3)x^3$, where x and y denote the measurements in miles. Find the rate of change of direction of a train when it is passing through the point (1, 1/3) on this track.

By differentiation of $y = (1/3)x^3$, we have $dy/dx = x^2$ and $d^2y/dx^2 = 2x$.

Substituting these in equation (1), we have

At (1, 1/3), $k = \frac{2}{2^{3/2}} = \frac{1}{\sqrt{2}}$ radian per mile (see equation (1)). This is the rate of change of direction of the train at the given point (1, 1/3). Actually, speed on transition curve = speed on circular curve.

Definitions

1. Cant or super elevation is the amount by which one rail is raised above the other rail. It is positive when the outer rail on a curved track is raised above inner rail and is negative when the inner rail on a curved track is raised above the outer rail.
2. Equilibrium speed is the speed at which the centrifugal force developed during the movement of the vehicle on a curved track is exactly balanced by the cant provided.

3. *Cant Deficiency* – Cant deficiency occurs when a train travels around a curve at a speed higher than the equilibrium speed. It is the difference between the theoretical cant required for such higher speed and the actual cant provided.
4. *Cant Excess* – Cant excess occurs when a train travels around a curve at a speed lower than the equilibrium speed. It is the difference between the actual cant and the theoretical cant required for such a lower speed.

Length of Transition Curve and Setting Out Transitions

1. The desirable length of transition L shall be maximum of the following three values:

   a. $L = 0.008\ C_a \times V_m$

   b. $L = 0.008\ C_a \times V_m$

   c. $L = 0.72\ C_a$,

   where L = the length of transition in metres.

   V_m = maximum permissible speed in kmph

   C_d = cant deficiency in millimetres.

   C_a = actual super elevation on curve in millimetres.

   The formulae (a) and (b) are based on rate of change of cant and deficiency of 35 mm per second. The formula (c) is based on the maximum cant gradient of 1 in 720 of 1.4 mm per metre.
2. For the purpose of designing future layouts of curve, future higher speeds (such as 160 km/h for Group A routes and 130 km/h for Group B routes) may be taken into account for calculating the length of transitions.
3. In exceptional cases where room is not available for providing sufficiently long transitions in accordance with the above, the length may be reduced to a minimum of 2/3 of the desirable length as worked out on the basis of formula (a) and (b) above, or 0.36 C_a (in metres) whichever is greater. This is based on the assumption that a rate of change of cant/ cant deficiency will not exceed 55 mm per second and the

maximum cant gradient will be limited to 2.8 mm per metre or 1 in 360. This relaxation shall apply to Broad Gauge only. For Narrow Gauge and Metre Gauge sections, cant gradient should not be steeper than 1 in 720. For Metre Gauge, the rate of change of cant/cant deficiency should not exceed 35 mm/second.

4. At locations where length of transition curve is restricted and, therefore, may be inadequate to permit the same maximum speed as calculated for the circular curve, it will be necessary to select a lower cant and/or a lower cant deficiency which will reduce the maximum speed on the circular curve but will increase the maximum speed on the transition curve. In such cases, the cant should be so selected as to permit the highest speed on the curve as a whole.

Application of Mathematics

Mathematics is used in almost all fields. Some of them are mentioned below:

1. astronaut,
2. astronomy,
3. astrology,
4. astrophysics,
5. physics,
6. statistics,
7. various surveys,
8. planning,
9. to find probability,
10. war field,
11. economics,
12. geography,
13. medical,
14. computers, etc.

Moreover, mathematical calculations are very useful from our birth to death; we entirely depend upon the various electronics and non-electronic equipment, which are farmed based on mathematics.

References

Charles, Lee Crandall, 1893, *The Transition Curve*, New York: J. Wiley and Sons.

Higgins, Arthur Lovat, 1922, *The Transition Spiral and Its Introduction to Railway Curves with Field Exercises in Construction an Alignment*, New York: Van Nostrand.

Little, Thomas, 2000, *A History of Greek Mathematics*, vol. 1: *From Thales to Euclid*, Chestnut Hill, MA: Health Adamant Media Corporation.

Rankine, Willam John Macquorn and W.J. Millar, 1883, A *Manual of Civil Engineering*, 17th edn, London: Charles Griffin.

Talbot, Arthur, Newell, 1904, *The Railway Transition Spiral*, New York: Engineering News Publishing.

38

History of Operations Research

J. Sengamalaselvi

Abstract: Operations Research (Operational Research, OR, or Management Science) includes a great deal of problem-solving techniques like mathematical models, statistics and algorithms to aid in decision making. Operations Research is employed to analyse complex real-world systems, generally with the objective of improving or optimizing performance. In other words, Operations Research is an interdisciplinary branch of applied mathematics and formal science which makes use of methods like mathematical modelling, algorithms statistics and statistics to reach optimal or near optimal solutions to complex situations. It is usually worried about optimizing the maxima (for instance, profit, assembly line performance, bandwidth, etc.) or minima (for instance, loss, risk, cost, etc.) of some objective function. Operational Research aids the management to accomplish its objectives utilizing scientific methods. The area of study known as the history of mathematics is primarily an investigation into the origin of discoveries in mathematics and, to a lesser extent, an investigation into the mathematical methods and notation of the past. Operations Research is a core course of many management majors. The aim of this paper is to present the history of Operations Research.

Keywords: Complex relationships, interdisciplinary approach, Operation Research, scientific method.

Introduction

It is generally agreed that Operations Research came into existence as a discipline during the Second World War when there was a critical need to manage scarce resources. However, a particular model and technique of Operations Research can be traced back as early as in the First World War, when Thomas Edison (1914-15) made an effort to use a tactical game board for finding a solution to minimize shipping losses from enemy submarines, instead of risking ships in actual war conditions about the same time A.K. Erlang, a Danish engineer, carried out experiments to study the fluctuations in demand for telephone facilities using automatic dialling equipment. Such experiments, later on, were used as the basis for the development of the waiting-line theory.

Since the war involved strategic and tactical problems that were highly complicated, to expect adequate solutions from individuals or specialists in a single discipline was unrealistic. Therefore, groups of individuals who were collectively considered specialists in mathematics, economics, statistics and probability theory, engineering, behavioural and physical sciences, were formed as special units within the armed forces, in order to deal with strategic and tactical problems of various military operations.

Such groups were first formed by the British Air Force and, later, the American armed forces formed similar groups. One of the groups in Britain came to be known as Blackett Circus. This group, under the leadership of P.M.S. Blackett was attached to the Radar Operational Research unit and was assigned the problem of analysing the coordination of radar equipment at gun sites. The effort of such groups, especially in the area of radar deduction, is still considered vital for Britain in winning the air battle. Following the success of the group, similar mixed-team approach was also adopted in other allied nations.

After the war was over, scientists who had been active in the military Operations Research groups made efforts to apply the Operations Research approach to civilian problems related to business, industry, research and development, etc. There are three important factors behind the rapid development of using the Operations Research approach.

i. The economic and industrial boom after the Second World War resulted in continuous mechanization, automation and decentralization of operations and division of management functions. This industrialization also resulted in complex managerial problems and, therefore, the application of operations research to managerial decision making became popular.

ii. Many operations researchers continued their research after the war. Consequently, some important advancements were made in various operations research techniques. In 1947, P.M.S. Blackett developed the concept of linear programming, the solutions of which are found by a method known as simplex method. Besides linear programming, many other techniques of Operations Research such as statistical quality control, dynamic programming, queuing theory and inventory theory were well-developed before the end of the 1950s.

iii. Greater analytical power was made available by high-speed computers. The use of computers made it possible to apply many Operations Research techniques for practical decision analysis.

During the 1950s, there was substantial progress in the application of Operations Research techniques for civilian activities along with a great interest in the professional development and education of Operations Research. Many colleges and universities introduced Operations Research in their curricula. These were generally schools of engineering, public administration, business management, applied mathematics, economics, computer science, etc. Today, however, service organizations such as banks, hospitals, libraries, airlines and railways, all recognize the usefulness of Operations Research in improving their work efficiency. In 1948, an Operations Research club was formed in England which later changed its name to the Operations Research Society of UK. Its journal, *OR Quarterly* first appeared in 1950. The Operations Research Society of America (ORSA) was founded in 1952 and its journal, *Operations Research* was first published in 1953. In the same year, The Institute of Management Sciences (TIMS) was founded as an international society to identify, extend and unify scientific knowledge pertaining to management. Its journal, *Management Science*, first appeared in 1954.

At the same point of time, R.S. Verma also set up an OR team at Defence Science Laboratory for solving problems of store, purchase and planning. In 1953, P.C. Mahalanobis established an Operations Research team in the Indian Statistical Institute, Kolkata for solving problems related to national planning and survey. The OR Society of India (ORSI) was founded in 1957 and it started publishing its journal *OPSEARCH* 1964 onward. In the same year, India along with Japan, became a member of the International Federation of Operational Research Societies (IFORS) with its headquarters in London. The other members of IFORS were UK, USA, France and West Germany.

A year later, project scheduling techniques – Program Evaluated and Review Technique (PERT) and Critical Path Method (CPM) – were developed as efficient tools for scheduling and monitoring lengthy, complex and expensive projects of that time. By the 1960s Operations Research groups were formed in several organizations. Educational and professional development programmes were expanded at all levels and certain firms, specializing in decision analysis, were also formed.

The American Institute for Decision Science came into existence in 1947. It was formed to promote, develop and apply quantitative approach to functional and behavioural problems of administration. It started publishing a journal, *Decision Science*, in 1970.

Because of Operations Research's multidisciplinary character and its application in varied fields, it has a bright future, provided people devoted to the study of Operations Research can help meet the needs of the society. Some of the problems in the area of hospital management, energy conservation, environmental pollution, etc. have been solved by Operations Research specialists. This is an indication of the fact Operations Research can also contribute towards the improvement of the social life and of areas of global need. However, in order to make the future of Operations Research brighter, its specialists have to make good use of the opportunities available to them.

Definitions of Operations Research

Because of the wide scope of application of Operations Research, giving its precise definition is actually difficult. However, a few definitions of Operations Research are as follows:

i. Operations Research is the application of the methods of science to complex problems in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of the system incorporating measurements of factors such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management in determining its policy and actions scientifically.

ii. The application of the scientific method to the study of operations of large complex organizations or activities. It provides top-level administrators with a quantitative basis for decisions that will increase the effectiveness of such organizations in carrying out their basic purpose.

Apart from being lengthy, the definition given by Operational Research Society of UK has been criticized because of the emphasis it places on complex problems and large system, leaving the reader with the impression that it is a highly technical approach suitable only to large organizations. The definition of OR Society of America contains an important reference to the allocation of scarce resources. The keywords used in the above definitions are scientific approach, scarce resources, system and model. The British definition contains no reference to optimization, while the American definition has no reference to the word, best. However, all two definitions point to the following characteristics of Operations Research:

i. Use of scientific method.

ii. Use of models to represent the complex relationships.

iii. Interdisciplinary approach.

iv. Provision of a quantitative basis for decision making.

A few other definitions, commonly used and widely acceptable, are as follows:

i. Operations Research is the systematic application of quantitative methods, techniques and tools to the analysis of problems involving the operations of systems.

ii. Operations Research is essentially a collection of mathematical techniques and tools, which in conjunction with a systems approach, are applied to solve practical decision problems of an economic or engineering nature.

These two definitions refer to the interdisciplinary nature of Operations Research. However, there is nothing that can stop one person from considering several aspects of the problem under consideration. Best definition of Operation Research is as follows:

* Operation Research, in the most general sense, can be characterized as the application of scientific methods, techniques and tools, to problems involving the operations of a system so as to provide those in control of the operations with solutions to the problems.

This above definition(*) refers to the military origin of the subject, where team of experts were not actually participating in military operations for winning the war but providing advisory and intellectual support for initiating strategic military actions.

This definition refers operations research as technique for selecting the best course of action out of the several courses of action available, in order to reach the desirable solution of the problem.

A few other definitions of Operations Research are as follows:

- Operations Research has been described as a method, an approach, a set of techniques, a team activity, a combination of many disciplines, an extension of particular disciplines (mathematics, engineering, economics), a new discipline, a vocation, even a religion. It is perhaps some of all these things.
- Operations Research may be described as a scientific approach to decision making that involves the operations of organizational system.
- Operations Research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.
- Operations Research is applied decision theory. It uses any scientific, mathematical or logical means to cope with the problems that confront the executive, when he tries to achieve a thoroughgoing rationality in dealing with his decision problems.
- Operations research is a scientific approach to problem-solving for executive management.

As the discipline of Operations Research grew, numerous names such as Operations Analysis, Systems Analysis, Decision Analysis, Management Science, Quantitative Analysis and Decision Science were given to it. This is because of the fact that the types of problems encountered are always concerned with effective decision, but the solution of these problems do not always involve research into operations or aspects of the science of management.

Features of Operations Research Approach

The board-based definition of Operations Research, with the additional features is as follows: Operations Research utilizes a planned approach following a scientific method and interdisciplinary team, in order to represent complex functional relationship as mathematical models, for the purpose of providing a quantitative basis for decision making and uncovering new problems for quantitative analysis. The board features of Operations Research approaches to any decision problem are summarized below:

INTERDISCIPLINARY APPROACH

Interdisciplinary approach for solving a problem of interdisciplinary teamwork is essential. This is because while attempting to solve a complex management problem, one person may not have the complete knowledge of all its aspects (such as economic, social, political, psychological, engineering, etc.). This means we should not expect one person to find a desirable solution to all managerial problems.) Therefore, a team of individuals specializing in mathematics, statistics, economics, engineering, computer science, psychology, etc. should be organized in a way that each aspect of the problem can be analysed by a particular specialist in that field. This would help to arrive to an appropriate and desirable solution of the problem. However, there are certain problem situations that can be analysed by even one individual.

SCIENTIFIC APPROACH

Scientific approach in Operations Research is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide those in control of operations with optimum solutions to the problems. The scientific method consists of observing and defining the problem; formulating and testing the hypothesis; and analysing the results of the test. The data so obtained is then used to decide whether the hypothesis should be accepted or not. If the hypothesis is accepted, the results should be implemented, otherwise, an alternative hypothesis has to be formulated.

HOLISTIC APPROACH

Holistic approach while arriving at a decision, an Operations Research team examines the relative importance of all conflicting and multiple objectives. It also examines the validity of claims of various departments of the organization from the perspective of its implications to the whole organization.

OBJECTIVE-ORIENTED APPROACH

An Operations Research approach seeks to obtain an optimal solution to the problem under analysis. For this, a measure of desirability (of effectiveness) is defined, based on the objective(s) of the organization. A measure of desirability so defined is then used to compare alternative courses of action with respect to their possible outcomes.

Operations Research Approach to Problem Solving

The most important feature of Operations Research is the use of the scientific method and the building of decision models. The Operations Research approach to problem solving is based on three phases, viz.:

i. judgement phase,
ii. research phase, and
iii. action phase.

JUDGEMENT PHASE

This phase includes:

a. Identification of the real-life problem.
b. Selection of an appropriate objective and the values related to this objective.
c. Application of the appropriate scale of measurement, that decides the measures of effectiveness (desirability).
d. Formulation of an appropriate model of the problem and the abstraction of the essential information, so that a solution to the decision-maker's goals can be obtained.

RESEARCH PHASE

This phase is the largest and longest amongst all the phases. However, even though the remaining two are not as long, they are also equally important as they provide the basis for a scientific method. This phase utilizes:

a. Observations and data collection for a better understanding of the problem.

b. Formulation of hypothesis and models.
c. Observation and experimentation to test the hypothesis on the basis of additional data.
d. Analysis of the available information and verification of the hypothesis using pre-established measures of desirability.
e. Prediction of various results from the hypothesis.
f. Generalization of the result and consideration of alternative methods.

ACTION PHASE

This phase consists of making recommendations for implementing the decision. This decision is implemented by an individual who is in a position to implement actions. This individual must be aware of the environment in which the problem occurred, be aware of the objective, of assumptions behind the problem and the required omissions of the model.

Conclusion

The Operations Research approach attempts to find global optimum by analysing interrelationships among the system components involved in the problem. One such situation is described below.

Consider the case of a large organization that has a number of management specialists but the organization is not exactly very well-coordinated. For example, its inability to properly deal with the basic problems of maintaining stocks of finished goods. To the marketing manager, stocks of a large variety of products are purely a means of supplying the company's customers with what they want and when they want it. Clearly, according to a marketing manager, a fully stocked warehouse is of prime importance to the company. But the production manager argues for long production runs, preferably on a smaller product range, particularly if a significant amount of time is lost when production is switched from one variety to another. The result would again be a tendency to increase the amount of stock carried but it is, of course, vital that the plant should be kept running. On the other hand, the finance manager sees stocks in terms of capital that is unproductively tied up and argues strongly for its reduction. Finally, there appears the personnel manager for whom a steady level of production is advantageous for having better labour relations. Thus, all these people would claim to uphold the interests of the organizations, but they do so only from their own specialized points of view. They may come up with contradictory solutions and obviously, all of them cannot be right. In view of this problem that involves the whole system, the decision maker, whatever his specialization, will need help. It is in the attempt to provide this assistance that Operations Research has been developed.

References

Abbas, A.E., 2014, "Teaching Decision-making with Social Networks'', *OR/MS Today*, 40(6).

Ackoff, R.L., 1967, "Management Misinformation Systems", *Management Science*, 14(4): 147-56. doi:10.1287/mnsc.l4.4.B147

Bowman, E. and R. Fetter, 1957, *Analysis for Production and Operations Management*, Homewood, IL: Richard D. Irvin.

De Souza, L.B., 2009, "Trends and Approaches in Lean Healthcare", *Leadership in Health Services*, 22(2): 121-39. doi:10.1108/7511870910953788

Heckmann, I., T. Comes and S. Nickel, 2015, "A Critical Review on Supply Chain Risk: Definition, Measure and Modeling", *Omega*, 52: 119-32. doi:10.1016/j.omega.2014.10.004

Radnor, Z.J., M. Holweg and J. Waring, 2012, "Lean in Healthcare: the Unfilled Promise", *Social Science & Medicine*, 74(3): 364-71. doi:10.1016/j.socscimed.2011.02.11

39

Graph Kernels in Protein Study
A Survey

D. Vijayalakshmi

Abstract: In recent research works, machine learning occupies an important place and has turned as an inevitable research discipline. The machine-learning methods analyses and extracts knowledge from available data and provides an easier way to understand the graph-structured data: proteins, protein–protein interaction, protein structures along chemical pathways, social networks, WorldWideWeb, programme flow. The prime objective of this paper is to present a survey of graph kernels in protein study, the special case of which includes kernels used in protein similarity study.

Keywords: Graph kernel, proteins, protein similarity study.

Introduction

SOCIAL networks, chemical pathways, protein structures and programme flow analysis can be represented as graph-structured data. The studies in these areas are developed by support vector machines. To analyse and study in these areas, there are many machine-learning methods. Among them support vector machine methods are more efficient. This forms a class of kernel methods which yields a more effective result when compared to existing methods. In this paper, some kernels, defined and used in the study of proteins, are discussed in brief.

Schlkopf and Smola (2002) initialize this kernel method. Kashima et al. (2003) introduce a kernel based on random walks on graphs and reduce the computation of kernel to solving a system of linear simultaneous equation. This kernel takes into account all label paths without computing feature values explicitly.

Kernel is defined based on all possible walks by Gartner et al. (2003). Here the computation is made easy by using product graphs: Based on this direct product kernel, non-contiguous graph kernels were introduced and the main advantage of these is the expressivity of their features space. He also proves that complete graph kernel computation is like deciding whether the given set of graphs is isomorphic or not.

Ramon and Gartner (2003) explain the method of computing the number of common sub-trees in two graphs. As we know, the sequence of label of vertices addresses a walk, every sub-tree pattern is addressed by a tree. Here the kernel counts the number of times the sub-trees of a tree pattern that occurs in given graphs. Let p be the root of sub-trees in the graph $G1$ and let q be the root of sub-trees in the graph $G2$. If the tree of height one is considered, then the kernel is defined by

$$K_{p,h,1} = \begin{cases} 1 \ \textit{if label } (p) = \textit{label } (q) \ h = 1. \\ 0 \ \textit{if label } (p) \neq \textit{label } (q) \ h = 1. \end{cases}$$

In the same way, sub-trees are considered for greater values of h.

Mahe and Vert (2009) talk about the family of graph kernels based on the common tree patterns in the graphs. Two kernels with explicit features of spaces and inner product are derived from Ramon and Gartner (2003). The complexity of the feature characterizing the graphs is minimized using the parameter λ. Mahe generalizes the walk-based kernel to a board class of kernels. He also defines tree pattern, tree pattern counting function, tree pattern graph kernel. The kernel is defined based on the pattern-counting function and this function returns a numerical value.

Mahe's kernel:

$$K(G,G') = \sum_{t \in T} p(t)\varphi_t(G), \varphi_t(G')$$

T set of trees

p: $T \rightarrow Z^+$ is a tree weighting function.

φ_t is tree pattern counting function.

Hovarth et al. (2004) proposes a graph kernel based on tree and cyclic pattern, irrespective of how frequent it appears in graph. For computing this cyclic-pattern kernel, possible cyclic and tree pattern from the graph is taken into account and intersection is applied. The result proves that the cyclic-pattern approach is faster than graph kernel based on frequent use of patterns.

Shervashidze et al. (2009) define graph kernels based on limited-size sub-graphs, i.e. graphs count all types of sub-graphs of three, four, five vertices. Using sampling method, the kernels are computed and these are applied to unlabelled graphs.

Shervashidze and Borgwaedt (2010) next defines a fast sub-trees kernel on graphs. He computes kernel for graphs with v nodes and e edges and maximum degree d and for sub-trees height t. He proves that this kernel can have a board application in bio-informatics for protein function prediction, etc.

A probem is called *NP* (Nondeterministic Polynomial) if its solution can be guessed and varified in polynomial time; nondeterministic means that no particular rule is followed to make the guess. A problem is *NP* hard if a method for solving it can be transformed into one for solving any *NP* problem. Borgwardt and Kriegel (2005) introduce kernel based on shortest path as considering all shortest paths and longest paths in a graph is *NP* hard. This kernel retains expressivity and is positive definite, i.e. the kernal proposed in paper "Borgardt, K.M. and H.P. Kriegel, 2005, "Shortest-path Kernels on Graphs", in Proceedings of the International Conference on Data Mining, pp. 74-81" can be translated or used for all paths in a graph.

Jiang et al. (2014) propose a simple, efficient and effective classification approach using graph kernel based on labels of graph structure. Initially, the protein structure is modelled as a graph based on amino acid sequence of protein. To the graph, kernel is applied to predict the function of protein. The result obtained classifies the protein according to its function.

Kashima et al. (2004) discuss about various kernel function based on vertex label and edge labels. Label sequence kernel is introduced using random walks on graphs. He further reduces the computation of the kernel as solving system of simultaneous linear equations. The kernel defined in this part has a promising application in a wide variety of problems in bioinformatics.

Thomas et al. (2009) introduces a substructure fingerprint kernel to identify the active sites of protein. In this part the protein is represented in terms of node labelled and edge weighted graphs.

Aasa et al. (2013) present graph hopper kernel between graphs. This kernel counts similar sub-path. The shortest paths between node pairs from the two graphs are compared by this kernel. The important fact is that the graph kernel can be decomposed into weighted sum of node kernels. This kernel is applied on graphs with any kind of node attributes. It is trivial that this kernel is a parameter-free kernel.

Costa and Grave (2010) introduce a neighbourhood sub-graph pairwise-distance kernel. This kernel depends on radius, distance and neighbourhood. $K(G, G') = (\sum_r \sum_d k_{r,d}(G, G')$. $K_{r,d}$ counts number of identical pairs of graphs of radius r and distance d.

Shervashidze et al. (2011) present a general definition of graph kernels that covers many known graph kernels. General graph kernel based on Weisfeiler Lehman graph kernel is described with example.

Matthias and Basak (2012) discuss the different types of kernel. The kernels are random walk kernel, shortest path kernel, tree pattern kernel, cyclic pattern kernel and graphlet kernel. Adding to this, he explains the application of these kernels in bioinformatics and cheminformatics.

Mathiew et al. (2008) give a brief review of Weisfeiler kernel. They introduce modified Weisfeiler kernel and prove the efficiency of kernel from the computation point of view.

Malinda (2013) represents protein surface as a graph based on contacts between amino acids in an innovative way. To study the similarity between graphs, he implements a shortest path kernel method. On applying this kernel he reaches 77.1 per cent accuracy. He used the result to predict the protein functional sites.

Giovanni et al. (2014) frame a new Weisfeiler–Lehman isomorphism test. This test consists of a new way of relabelling phase. From different ways of relabelling, they derive two kernels that compute faster than the existing kernels.

Lixiang et al. (2015) define a mixed Weisfeiler–Lehman graph kernel based on Weisfeiler–Lehman test of isomorphism. This mixed kernel is applied to Weisfeiler–Lehman graph. The main advantage of this kernel is, it enhances accurate classification.

Markus (2008) introduces graph kernel based on labelled walk. This kernel is constructed based on structural information of graph.

Wenchao et al. (2016) identify the similarity between programmes using mixed Weisfeiler–Lehman graph kernel. The similarity is measured in the way the programmes call the set of function on execution of the programme. As similar programmes have similar way of data flow, the similarity is measured in this way.

Mahe and Vert (2009), based on the sub-trees kernel defined by Ramon and Gartner (2003), propose two kernel functions with description of their feature spaces. A parameter is introduced, which increases the complexity of the sub-tree used. On decreasing the parameter, the kernel is the classical walk-based kernel. The formulation of this kernel initially allows generalizing the associated feature space of the sub-trees removed.

A kernel is defined to predict protein–protein interaction in (Ben-Hur and Noble 2005). This kernel uses data from protein sequences, gene ontology annotation and properties of network to predict the interaction.

Benoit et al. (2004) introduce a tree let kernel. This tree let kernel depends on cyclic information. Topological information is encoded and each tree let is assigned a weight which makes computation easier. This kernel can be computed with relevant complexity based on cyclic pattern.

Vishwanath et al. (2010) theoretically show that the existing kernels defined between graphs are related to each other. Second, he introduces new algorithms for efficient computation of these kernels.

Marco et al. (2012) propose a new method to predict protein function from protein structure. Implementing hierarchical clustering on protein backbone, graph for each protein is constructed. Next to this a shortest path kernel is defined to measure similarity between graphs.

Kernel methods provide an efficient way to measure similarity/dissimilarity between proteins. In protein study, the path length between each pair of vertices in protein graph and the secondary structure of the protein can be used to frame the kernel. Kernel function can be defined based on the neighbourhood of vertices of protein graph. These are some ways of framing kernel for the protein graph to measure similarity/dissimilarity between proteins.

Conclusion

It is trivial that the kernel methods are easier method and occupies an inevitable role in the study of protein especially in similarity study of proteins. Besides its simplicity the kernel method provides a good result when compared with existing methods. At last the kernel methods transforms even complex problems to simple one.

References

Borgardt, K.M. and H.P. Kriegel, 2005, "Shortest-path Kernels on Graphs", in *Proceedings of the International Conference on Data Mining*, pp. 74-81.

Ben-Hur A and W.S. Noble, 2005, "Kernel Methods for Predicting Protein-Protein Interactions", *Bioinformatics*, 21: 38-46.

Costa F. and K. De. Grave, 2010, "Fast Neighborhood Sub-graph Pair Wise Distance Kernel", in *Proceeding of International Conference on Machine Learning*, pp. 255-62.

Feragen, Aasa, Niklas Kasenberg, Jens Petersen, Marleen de Bruijne and Karsten Borgwardt, 2013, "Scalable Kernels for Graphs with Continuous Attributes", in *Neural Information Processing System.* (https://www.researchgate.net/publication/286043077_Scalable_kernels_for_graphs_with_continuous_attributes).

Gartner, T. P.A., Flach and S. Wrobel, 2003, "On Graph Kernels: Hardness Results and Efficient Alternatives", in *Proceedings of the Annual Conference on Computational Learning Theory*, pp. 129-43. (https://www.researchgate.net/publication/221497506_On_Graph_Kernels_Hardness_Results_and_Efficient_Alternatives)

Gatizere, Benoit and Luc Brun, 2004, *Graph Kenels Based on Relevant Patterns and Cycle Information for Chemo Informatics*. (https://www.researchgate.net/publication/234025117_Graph_Kernels_Based_on_Relevant_Patterns_and_Cycle_Information_for_Chemoinformatics)

Giovanni, Da San Martino, Nicolò Navarin and Alessandro Sperduti, 2014, "Graph Kernels Exploiting Weisfeiler–Lehman Graph Isomorphism Test Extension", *Neural Information Processing*, vol. 8835 of the series Lecture Notes in Computer Science, pp. 93-100.

Horvath, T., T. Gartner and S. Wrobel, 2004, "Cyclic Pattern Kernels for Predictive Graph Mining", in *Proceedings of the ACM SIGKDD Conference on Knowledge Discovery and Data Mining*, pp. 158-67.

Kashima, H., K. Tsuda and A. Inokuchi, 2003, "Marginalized Kernels between Labeled Graphs", in *Proceeding of the International Conference on Machine Learning* , Washington DC, USA.

Kashima, H., K. Tsuda and A. Inokuchi, 2004, "Kernels for Graphs", *Kernel Methods in Computational Biology*, 39(1): 101-13.

Lixiang Xu, Jin Xie, Xiaofeng Wang and Bin Luo, 2015, "A Mixed Weisfeiler–Lehman Graph Kernel", *Graph-Based Representations in Pattern Recognition,* vol. 9069 of the series Lecture Notes in Computer Science, pp. 242-51.

Mahe, P. and J.P.Vert, 2009, "Graph Kernels Based on Tree Patterns for Molecules", *Machine Learning*, 75(1): 3-35.

Mahe Pierre and Jean-Philippe Vert, 2009, "Graph Kernels Based on Tree Patterns for Molecules", *Machine Learning*, 75(1): 3-35.

Matthias, Dehmer and Subhas C. Basak, 2012, "Statistical and Machine Learning Approaches for Network Analysis", Wiley Series in Computational Statistics.

Mathiew, Senelle, 2008, "Measure of Similarity between Graphs from Similarity to Density", from: Fouss, François, 24/01/2014. (http://hdl.handle.net/2078.1/161671)

Malinda, Vikum and Sanjaka Benaragama Vidanelage, 2013, Paper submitted for Master of Science degree.

Marco, A. Alvarez and Changhui Yan, 2012, "A New Protein Graph Model for Function Prediction", *Computational Biology and Chemistry* 37: 6-10 (https://www.sciencedirect.com/science/article/abs/pii/S1476927112000047?via%3Dihub).

Markus, Heinonen, 2008, "Graph Kernel" (https:// www.cs.helsinki.fi /group /smart/teaching/58308109/slides_Markus.pdf).

Qiangrong, Jiang, Xiongzhikang and Zhai Can, 2014, "Graph Kernels and Applications in Protein Classification", *Journal of Chemical and Pharmaceutical Research*, 6(2): 563-69.

Ramon J. and T. Gartner, 2003, "Expressivity versus Efficiency of Graph Kernels", technical report, First International Workshop on Mining Graphs, Trees and Sequences.

Schlkopf, B. and A.J. Smola, 2002, *Learning with Kernels*, Cambridge, MA: MIT Press (https://www.cs.utah.edu/~piyush/teaching/learning-with-kernels.pdf).

Shervashidze, N., S.V.N. Vishwanathan, T. Petri, K. Melhorn and K.M. Borgwardt, 2009, "Efficient Graphlet Kernels for Large Graph Comparisons", in *Artificial Intelligence and Statistics* (https://proceedings.mlr.press/v5/shervashidze09a/shervashidze09a.pdf).

Shervashidze N. and K.M. Borgwaedt, 2010, "Fast Sub-trees Kernels on Graphs", in *Advances in Neural Information Processing Systems 22, 23rd Annual Conference on Neural Information Processing Systems 1660-68* (https://proceedings.neurips.cc/paper/2009/file/0a49e3c3a03ebde64f85c0bacd8a08e2-Paper.pdf).

Shervashidze, N., P. Schweiter, E.J.V. Leeumen, K. Mehlhorn and K.M. Borgwardt, 2011, "Weisfeiler–Lehman Graph Kernels", *Journal of Machine Learning Research*, 12: 2539-61 (https://www.jmlr.org/papers/volume12/shervashidze11a/shervashidze11a.pdf).

Thomas, Fober, Maeco Mernberger, Ralph Morittz and Eyke Hullermeier, 2009, "Graph Kernels for the Comparative Analysis of Protein Active Sites", German Conference on Bioinformatics, Halle, Germany (https://www.researchgate.net/publication/221493505_Graph-Kernels_for_the_Comparative_Analysis_of_Protein_Active_Sites).

Vichy, S., N. Vishwanathan, N.N. Schraudolph, R. Kondor and K.M Borgwardt, 2010, "Graph Kernels" *The Journal of Machine Learning Research*, 11: 1201-42.

Wenchao Li, Hassen Saidi, Huascar Sanchez, Martin Schäf and Pascal Schweitzer, 2016, "Detecting Similar Programs via the Weisfeiler–Lehman Graph Kernel", *Software Reuse: Bridging with Social-Awareness* vol. 9679 of the series Lecture Notes in Computer Science, pp. 315-30 (https://www.martinschaef.de/papers/icsr2016.pdf).

40

Spectral Techniques in Protein Study

A Survey

D. Vijayalakshmi
K. Divya

Abstract: Network science is a vast multidisciplinary field occuping a prominent position in present-day research. The techniques from spectral graph theory, probability, approximation theory plays major role in network science. We give a survey of graph spectral techniques used in protein study. This survey consist of description of methods of graph spectra used in different study area of protein like protein domain decomposition, protein function prediction, similarity.

Keywords: Eigenvalues, spectra, protein.

Introduction

SPECTRAL graph theory is becoming an important, unavoidable part in recent researches. Spectral graph theory narrates the relation between the graph and its eigenvalues. For example, the connectivity of graph is defined by Laplacian matrix, the number of bipartite connected components is obtained from signless Laplacian matrix and many more properties are revealed using these techniques. In this paper, we brief the various spectral techniques used in protein study.

In Peng and Tsay (2014), the proteins are represented as graphs. The Laplacian matrix for the graph and its eigenvalues are considered. The similarity between proteins is measured using Euclidean distance of Laplacian spectra. The important fact discussed in this journal is the stability of the graph constructed for the protein and its stability is verified using entropy.

In Banerjee (2012) the normalized Laplacian of biological network graph is utilized. The information that the normalized spectrum can have is investigated based on the eigenvalues. The empirical networks are classified based on their properties detected through spectral plots of Laplacian matrix.

Do Phuc and Nguyen Thikim Phung (2009) present protein structure graphically. Jacobi rotation algorithm is obtained by spectrum of normalized Laplacian. The Euclidean distance between these spectra are used to measure similarity between proteins. M-Tree method is used to index the spectral vector and this increases the efficiency of similarity search in protein structure graph database.

Dragos (2012) discusses various spectra of graphs. The similarity between the proteins using spectral distance is explained in detail i.e. if the spectral distance is small, the graphs are similar; if zero, there are co-spectral and if the distance is high the graphs are dissimilar.

In Vijayalakshmi and Rao (2017), proteins are presented as a graph and the degree distance matrix of the protein graph is obtained. The least positive eigenvalue of the degree distance matrix is considered as a parameter to measure similarity between proteins. Based on the distance between the least positive eigenvalues, the percentage of similarity is obtained.

Peng and Tsay (2014) and Dragos (2012) study the protein structural similarity using the spectra of adjacency matrix, Laplacian matrix, signless Laplacian matrix and Seidal adjacency matrix. The similarity, measured based on the Euclidean distance between the spectra of Seidal adjacency matrix, yields a better result.

Peng and Tsay measure the stability of the graph constructed for proteins. Yan Yan (2011) describes several methods in solving protein structure identification by graph theory is the topic of study in the paper. He first introduced the development of protein structure identification and existing problem. Identification of side chain clusters in protein structure is done by spectral method.

Clusters are obtained from the second lowest eigenvalue and its vector of Laplacian matrix. Side chain that makes larger number of interaction in a cluster is obtained from top eigenvalue and its vector.

Tuan D. Pham (2006) studies similarity of protein sequence and calculated similarity using spectral approaches. Linear predictive coding [LPC] of protein sequences, based on Electron–Ion interaction potential, is done in the paper.

This method reveals non-trivial results in the study of functionally related protein sequence and functionally non-related protein sequence.

These existing methods provide a right direction in the research of protein study using matrices. It gives a clear idea of constructing new matrices relevant to the researches undergone. Irrespective of the research problem, the problem can be modelled as a graph satisfying the constraints in the problem. After converting to graph, a novel matrix or an existing matrix can be associated with the graph that makes the study easier.

This way of approach reduces the complex problem to a simple one carrying all the constraints specified in the statement of problem.

Conclusion

Spectral graph theory plays a vital role in protein study. These techniques carry all information about the matrix and these matrices carry all information about the graph and thus the protein. The methods appear to be simpler, but they are efficient.

References

Banerjee, Anirban, 2008, "The Spectrum of the Graph Laplacian as a Tool for Analysing Structure and Evolution of Networks", dissertation. (https://www.iiserkol.ac.in/~anirban.banerjee/Banerjee_PhD_Thesis.pdf)

Dragos, Cvetkovie, 2012, "Spectral Recognition of Graphs", *Yugoslav Journal of Operation Research*, 22, 2 November/15-16/DOI:10.2298/yJOR120925025c.

Peng, Sheng-Lung and Yu Wei Tsay, 2014, "Adjusting Protein Graphs Based on Graphs Entropy", *BMC Bio Informatics,* (suppl 15): S6. (http//www.biomedcentral.com 1471-2105/15/S15/S6)

Peng, Sheng-Lung and Yu-Wei Tsay, 2012, "On the Usage of Graph Spectra in Protein Structural Similarity". (http://www.researchgate.net./publication/266500209_on_the_usage_of_graph_spectra_in_Protein_structureal_similarity).

Pham, Than D., 2006, "Spectral Analysis of Protein Sequences", http://www.semanticscholar.org.

Phuc, Do and Nguyen Thikim, 2009, "Spectral Vectors and M-Tree for Graphs Clustering and Searching in Graph Databases of Protein Structures", *International Journal of Computer and Information Engineering*, 3(8).

Vijayalakshmi, D. and K. Srinivasa Rao, 2017, "DD Matrix with Least Positive Eigen Value and Protein Similarity", *International Journal of Pure and Applied Mathematics*, 115(9): 331-42.

Yan, Yan, Shanggui Zhong and Faug-xiangwu, 2019, "Application of Graph Theory in Protein Structure Identification", from International Workshop on Computational Proteome, December 2010, *Proteome Science*, 9 (Supp. 1): S17, http://www.proteomesci.com/content/9/S1/S17.

41

Review of Wiener Index and Its Applications

A. Dhanalakshmi
V. Kasthuri

Abstract: The topological index is a numeric quantity associated with chemical organization purpose and shows the correlation of chemical structures with many physico-chemical properties. Wiener Index, which defines the sum of distances between all unordered pairs of vertices in a graph, is one of the most popular topological indices. In this paper, we have discussed review of the Wiener Index and how it is applied to various fields.

Keywords: Hosoya polynomial, molecular descriptors, topological indices, Wiener index.

Introduction

CHEMICAL graph theory is the topology branch of mathematical chemistry which applies graph theory to mathematical modelling of chemical phenomena. Molecular descriptors play a fundamental role in chemistry, pharmaceutical sciences, environmental protection policy and health researches, as well as in quality control, being the way molecules, thought of as real bodies, are transformed into numbers, allowing some mathematical treatment of the chemical information contained in the molecule. In the fields of chemical graph theory, molecular topology and mathematical chemistry, a topological index, also known as a connectivity index, is a type of a molecular descriptor that is calculated based on the molecular graph of a chemical compound. In chemical graph theory, the Wiener index introduced by Harry Wiener, is a topological index of a molecule, defined as the sum of the lengths of the shortest paths between all pairs of vertices in the non-hydrogen atoms in the molecule. In the fields of chemical graph theory, molecular topology and mathematical chemistry, a topological index, also known as a connectivity index, is a type of a molecular descriptor that is calculated based on the molecular graph of a chemical compound.

Topological Indices

Topological indices are numerical parameters of a graph which characterize its topology and are usually graph invariant. Topological indices are used, for example, in the development of quantitative structure–activity relationships (QSARs) in which the biological activity or other properties of molecules are correlated with their chemical structure. Topological descriptors are derived from hydrogen-suppressed molecular graphs, in which the atoms are represented by vertices and the bonds by edges. The connections between the atoms can be described by various types of topological matrices (e.g. distance or adjacency matrices), which can be mathematically manipulated so as to derive a single number, usually known as graph invariant, graph-theoretical index or topological index. As a result, the topological index can be defined as two-dimensional descriptors that can be easily calculated from the molecular graphs, and do not depend on the way the graph is depicted or labelled and there is no need of energy minimization of the chemical structure.

Mircea V. Diudea and Ivan Gutman (1998) obtained the undefined approach to the Wiener topological index and its various recent modifications. He named Wiener index or Wiener number and it was introduced for the first time. (Note that in the great majority of chemical publications dealing with the Wiener number, it is denoted by W. Nevertheless, in this paper we use the symbol *We* in order to distinguish between the Wiener index and other Wiener-type indices.) Using the language which in theoretical chemistry emerged several decades after Wiener, we may say that *We* was conceived as the sum of distances between all pairs of vertices in themolecular graph of an alkane, with the evident aim to provide a measure of the compactness of the respective hydrocarbon molecule.

In 1947 and 1948, Wiener published a series of papers showing that there are excellent correlations between *We* and a variety of physico-chemical properties of organic compounds. Nevertheless, progress in this field of research was by no means fast. It took some fifteen years until Stiel. Since 1976, the Wiener number has found a remarkable variety of chemical applications. Thodos became the first scientists apart from Wiener to use *We*. Only in 1971 Hosoya gave a correct and generally applicable definition of *We*. In 1975/76 Rouvaray and Crafford reinvented *We*, which shows that even at that time the Wiener-number-concept was not widely known among theoretical and mathematical chemists. In molecular graph, Mircea V. Diudea uses:

$$W_e = W_e(G) = \sum_{x<y} D_{xy}.$$

Finally, somewhere in the middle of the 1970s, Wiener index began to rapidly gain popularity, resulting in scores of published papers. In the 1990s, we were witnesses of another phenomenon: a large number of other topological indices have been put forward, all being based on the distances between vertices of molecular graphs and all being closely related to Wiener number. The aim of this article is to provide an introduction to the theory of the Wiener index and a systematic survey of various Wiener-type topological indices and their interrelations. In order to achieve this goal, we first need to remind the readers of a few elementary facts of the chemical graph theory.

$$W_e = W_e\left(G\right) = \sum{}_e N_{i,e} N_{j,e}$$
$$W_p = W_p\left(G\right) = \sum{}_p N_{i,p} N_{j,p}$$

Mohammed Salaheldeen Abdelgader et al. (2018). computated the topological indices of Some Special graphs mathematics and explained the study of molecular structures, represents an interdisciplinary science called chemical graph theory or molecular topology. By using tools taken from graph theory, set theory and statistics, we attempt to identify structural features involved in structure–property activity relationships. Molecules and modelling unknown structures can be classified by the topological characterization of chemical structures with desired properties. Much research has been conducted in this area in the last few decades. Also they developed the oldest degree-based topological index, the Randi'c index. The degree-based topological indices, the atom-bond connectivity (ABC) and geometric–arithmetic (GA) indices, are of great importance, with a significant role in chemical graph theory, particularly in chemistry. Precisely, a topological index Top(G) of a graph is a number such that, if H is isomorphic to G, then Top(H) = Top(G). It is clear that the numbers of edges and vertices of a graph are topological indices. We let G = (V, E) be a simple graph, where $V(G)$ denotes its vertex set and $E(G)$ denotes its edge set. For any vertex $u \in V(G)$, we call the set $NG(u) = fv \in V(G)juv \in E(G)g$ the open neighbourhood of u; we denote by du the degree of vertex u and by Su = $\mathring{a}v \in NG(u)$ $d(v)$ the degree sum of the neighbours of u. The number of vertices and number of edges of the graph G are denoted by ($V(G)$) and ($E(G)$), respectively. A simple graph of order n in which each pair of distinct vertices is adjacent is called a complete graph and is denoted by Kn. The notation in this paper is taken from the books. In this paper, we study the molecular topological properties of some special graphs: Cayley trees, $G2n$; square lattices, SLn; a graph Gn; and a complete bipartite graph, Km, n. Additionally, the indices (ABC), ($ABC4$), (GA) and ($GA5$) of these special graphs, whose definitions are discussed in the materials and methods section, are computed.

The concept of topological indices came from Wiener while he was working on the boiling point of paraffin and was named the index path number. Later, the path number was renamed as the Wiener Index. Hayat and Imran (2014) studied various degree-based topological indices for certain types of networks, such as silicates, hexagonals, honeycombs and oxides. Hayat and

Imran (2014) studied the molecular topological properties and determined the analytical closed formula of the ABC, $ABC4$, $ABC5$, GA, $GA4$ and $GA5$ indices of Sierpinski networks. M. Darafsheh (2010) developed different methods to calculate the Wiener Index, Szeged Index and Padmakar–Ivan Index for various graphs using the group of automorphisms of G. He also found the Wiener indices of a few graphs using inductive methods. A. Ayache and A. Alameri (2016) provided some topological indices of mk-graphs, such as the Wiener Index, the hyper-Wiener Index, the Wiener polarity, Zagreb indices, Schultz and modified Schultz indices and the Wiener-type invariant.

A. Behtoei et al. (2011) determined new inequalities for Wiener and hyper Wiener indices, in term of the first and the second Zagreb indices and the number of hexagons in these graphs. These inequalities improve the bounds obtained by Gutman and Zhou and are the best possible bounds. Our notations are standard and mainly taken from Alameri, A. et al (2006). Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We denote by $d(x, y)$, $N(x)$ and $d(x)$, the distance between vertices x and y, vertices of distance one with vertex x and the degree of x, respectively. Also for each $e = uv \in E(G)$ we use the notations $Ne(v)$, $ne(v)$ and $\alpha e(v)$ for the set of vertices $t \in V(G)$ with $d(v, t) < d(u, t)$, $|Ne(v)|$ and $t \in Ne(v)\ d(t)$, respectively. A topological index is a number related to a graph which is a structural invariant, i.e. it is fixed under graph automorphisms. The Wiener Index, denoted by W, is defined as the sum of all distances between vertices of a graph.

Having a molecule, if we represent atoms by vertices and bonds by edges, we obtain a molecular graph (Martin Knor, 2016. Graph theoretic invariants of molecular graphs, which predict properties of the corresponding molecule, are known as topological indices. The oldest topological index is the Wiener Index, which was introduced in 1947 as the path number. Martin Knor obtained some fundamental property of Wiener Index:

$$N_2(F) = \sum_{l<i<j<p} n(T_i)n(T_j).$$

The Wiener Index (i.e. the total distance or the transmission number), defined as the sum of distances between all unordered pairs of vertices in a graph, is one of the most popular molecular descriptors. In this article we summarize some results, conjectures and problems on this molecular descriptor, with emphasis on works we were involved in. At first, the Wiener Index was used for predicting the boiling points of paraffins, but later a strong correlation between the Wiener index and the chemical properties.

A representation of an object giving information only about the number of elements composing it and their connectivity is named as topological representation of an object. A topological representation of a molecule is called a molecular graph. A molecular graph is a collection of points representing the atoms in the molecule and a set of lines representing the covalent bonds. These points are named vertices and the lines are named edges in graph theory language. J. Baskar Babujee and S. Ramakrishnan (2012) introduce new topological indices which yield the Wiener, hyper-Wiener, Schultz and modified Schultz indices as special cases for trees. One advantage of this method is that in computing Schultz and modified Schultz indices of trees we need not take into account the distances between vertices. The advantage of topological indices is that they may be used directly as simple numerical descriptors in comparison with physical, chemical or biological parameters of molecules in Quantitative Structure Property Relationships (QSPR) and in Quantitative Structure Activity Relationships (QSAR). One of the most widely known topological descriptors is the Wiener Index named after chemist Harold Wiener. Wiener Index correlates well with many physico-chemical properties of organic compounds and as such has been well studied over the last quarter of 20th century.

Zagreb group indices $M1(G)$ and $M2(G)$ appeared in the topological formula for the π-electron energy of conjugated systems. Recently introduced Zagreb co-indices are dependent on the degrees of non-adjacent vertices and thereby quantifying a possible influence of remote pairs of vertices to the molecule's properties. Platt number was used to predict the physical parameters of Alkanes. Reverse Wiener Index is used to produce QSPR models for the alkane molar heat capacity.

J. Baskar Babujee and Ramakrishnan (2012) investigated few topological indices like Wiener index, Zagreb index, Zagreb coindex, Platt number, geometric – arithmetic index and reverse Wiener index for graphs. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. We denote by $d(x, y)$, $N(x)$ and $d(x)$, the distance between vertices x and y, vertices of distance one with vertex x and the degree of x, respectively. Also for each $e = uv \in E(G)$ we use the notations $Ne(v)$, $ne(v)$ and $\alpha e(v)$ for the set of vertices $t \in V(G)$ with $d(v, t) < d(u, t)$, $|Ne(v)|$ and $t \in Ne(v)$ $d(t)$, respectively. A topological index is a number related to a graph which is structurall invariant, i.e. it is fixed under graph automorphisms. The Wiener Index, denoted by W, is defined as the sum of all distances between vertices of a graph.

References

Abdelgader, Mohammed Salaheldeen, Chunxiang Wang and Sarra Abdalrhman Mohammed, 2018, "Computation of Topological Indices of Some Special Graphs Mathematics", 6(3): 33.

Alameri, A., A. Modabish and A. Ayache, 2016, "New Product Operation on Graphs and It's Properties", *International Mathematical Forum*, vol. 11(8): 357-68.

Babujee, B. and S. Ramakrishna, 2012, "Topological Indices and New Graph Structures", *Applied Mathematical Sciences* 6(108): 5383-5401.

Baskar, Babujee J. and Ramakrishnan, 2012, "Applied Mathematical Sciences", Topological Indices and Graph Structures, vol. 6 (108): 5383-5401.

Behtoei, A., M. Jannesari and B. Taeri, 2011, "Communications in Mathematical and in Computer Chemistry", *Commun. Math. Comput. Chem.*, 65: 71.

Darafsheh, M., 2010, "Computation of Topological Indices of Some Graphs", *Acta Appl. Math.*, 110: 1225-35.

Diudea, Mircea V. and Ivan Gutman, 1998, "Wiener Type Topological Indices", *Croatica Chemica Acta*, 71(1): 21-51.

Gao, W., Y. Chen and W. Wang, 2017, "The Topological Variable Computation for a Special Type of Cycloalkanes", *J. Chem*. doi:10.1155/2017/6534758.

Hayat, S. and M. Imran, 2014, "Computation of Topological Indices of Certain Networks", *Appl. Math. Comput*., 240: 213-28.

Knor, Martin, Riste Skrekovski, Aleksandra Tepeh, 2015, "Mathematical Aspects of Wiener Index", *Ars Mathematica Contemporanea*, 11(2).

Wiener H., 1947, *J. Am. Chem. Soc*., 69: 17-20; *J. Am. Chem. Soc*., 69: 2636-38.

42

MATLAB in Protein Study

D. Vijayalakshmi
A. Shakila

Abstract: The study in bioinformatics involves typical database search of DNA, RNA or protein. Based on the way of search the required studies are further developed one such way are MATLAB and its programmes. In this paper, we brief about the use of MATLAB in various studies of proteins encode, amount of protein adsorption on particle, sequence alignments, protein structure tessellations which help in making the studies easy.

Keywords: Sequence alignment, protein, MATLAB.

Introduction

A SEQUENCE alignment is regulating the biological sequences including DNA (deoxyribonucleic acid) or RNA (ribonucleic acid) or protein.The study about sequence alignment can be done in many ways and one such way is using software. These studies are useful in identifying the similarity between proteins which make the protein studies simpler. Similarity of protein is identifying the degree of similarity between two sequences. Even though the proteins do not show common function based on structures, sequence alignment is one of the powerful methods to identify the structure and function of a protein. Sequence alignment is used to identify regions of similarity between two biological sequences (protein or nucleic acid), this type of alignment is based on numerical values. Nowadays researchers use computer-based language (MATLAB) to simplify the method of identifying the similarity of protein sequence. In this paper we brief about some MATLAB methods used in protein study.

Researcher Wen Zhang and MengKe (2014) analyses protein sequences using MATLAB toolbox (named Protein Encoding), which helps to represent or encode protein sequences for bioinformatics. Researcher Meghna Mathur and Geetika (2013) discuss various sequence alignment methods using Needleman Wunsch and Smith Waterman algorithms in MATLAB. Researcher Majid Masso (2010) discusses Tessellation of protein structure by the atomic coordinates in 3-D using MATLAB. Researcher Asavari Mehta (2014) developed a MATLAB model that will estimate the amount of blood plasma protein that will adsorb to the surface of a nanoparticle used in targets.

Protein Sequence Encoding

In these Wen Zhang and MengKe describe the protein encoding

Table 42.1: Features and Length of Proteins

Features	*Length*
Amino acid composition	20
Dipeptide composition	400
Moreau-Broto autocorrelation	8*nlag
Moran autocorrelation	8*nlag
Geary autocorrelation	8*nlag
CTDC	21
CTDT	21
CTDD	105
Conjoint triad	343
Sequence-order-coupling number	2*nlag
Quasi-sequence-order	40 + 2*nlag
Pseudoamino acid composition	20 + nlag
Amphiphilic pseudo amino acid composition	20 + 2*nlag
Amino acid pair	400
Binary profile	20*N

which are used for identifying the bioinformatics in protein sequence using MATLAB. These features and its length are shown in Table 42.1.

Numerical Representation Using MATLAB

The toolbox consists of four windows: input, result, descriptors and buttons. The input block is used to enter sequences, and a sequence is in the fasta format. The output blocks resulting numerical vectors. The descriptors panel having various descriptors and the users were able to choose features in the panel in an easier way.

The four buttons used for: run(seq), run(file), save, exit. The first and second buttons are used to start the encoding procedure. The third button is used for result. The last button closes the dialogue box. By using the toolbox we can easily get the protein sequences into the numerical values and use them to predict the protein functions or structures. Wen Zhang and MengKe discuss a MATLAB toolbox (protein encoding), which helps to represent or encode protein sequences as numerical vectors for bioinformatics. This MATLAB toolbox is easy to use, and users without the computer science background can easily understand the sequence of protein.

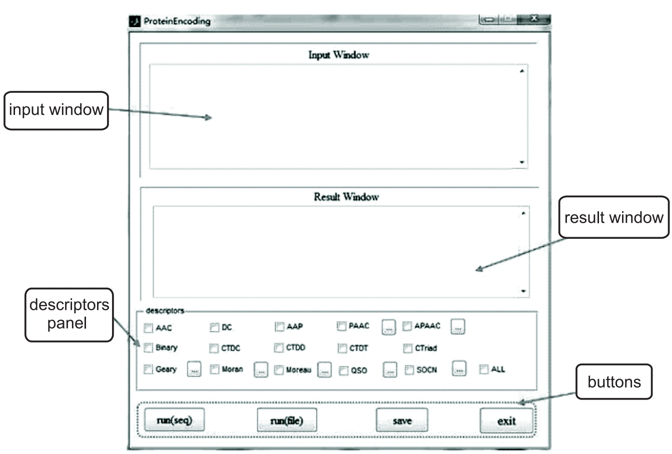


fig. 42.1: MATLAB toolbox for protein encoding

Methods of Alignment

DOT MATRIX METHOD

A dot matrix analysis is primarily a method for comparing two sequences to align the characters between the sequences. It is used to locate the regions of similarity between two sequences. Similar structure shows similar evolution, which provides information about the functions of these sequences (*fig*. 42.2).

The dot matrix plot is created by designating one sequence on the horizontal axis and designating the second sequence on the vertical axis of the matrix. Diagonal lines within the matrix indicate regions of similarity. The dot matrix computer programs do not show an actual alignment.

DYNAMIC PROGRAMMING

Dynamic programming (DP) algorithms are used for complex problems. DP algorithm has some problem with the following key points:

1. It should have an optimal substructure.
2. It must contain overlapping sub-problems.

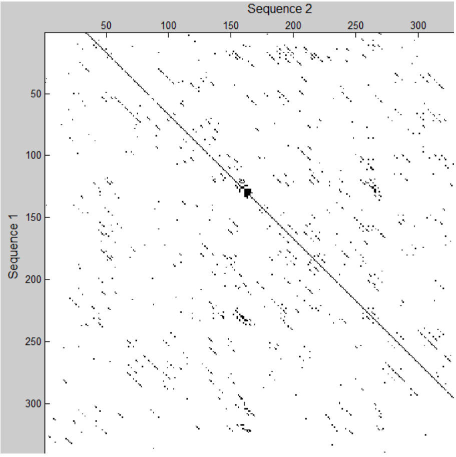


fig. 42.2: Sequence dot plot between Russian neanderthal and German neanderthal

DP works by first solving every sub-problem just once, and saves its answer to avoid the work of recalculating the answer every time, the sub-problem is encountered. Each intermediate answer is stored with a score, and DP finally chooses the sequence of solution that have the highest score. Both global and local alignments may be made by simple changes in DP algorithm.

Scoring functions – example

w (match) = – 2 or substitution matrix

w (mismatch) = – 1 or substitution matrix

w (gap) = – 3.

Dynamic programming has an alignment for a given set of scoring function which is its advantage. But it is slow because of the large number of steps and memory requirement which increases as the square of the sequence lengths. Dynamic programming has two algorithms that are used sequence alignment Needleman Wunsch and Smith Waterman algorithms.

SEQUENCE ALIGNMENT TOOLS

Meghna Mathur and Geetika discuss the Local Basic (BLAST), Alignment Search Tool, which is an algorithm for comparing sequence information, such as the amino-acid sequences of different proteins. It creates the fundamental problem and the heuristic algorithm is used for alignment. They, using a heuristic method, BLAST, finds similar sequences, not by comparing two sequence fully, but simple matches between the two sequences. A sequence can be evaluated based on various factors like algorithm, probability, accuracy and definiteness of an algorithm.

1. The algorithm that takes less time to identify sequence.
2. Probability, it helps to obtain accurate results and higher speed.
3. The factor can be accuracy of an existing algorithm. An algorithm should always give one output to the number of inputs applied and accuracy can be defined.
4. The factor can be definiteness. Definiteness means the algorithm should have finite number of steps.

If an algorithm does not have finite number of steps then the algorithm cannot give the correct results. Multiple sequence alignment has emerged to have a lot of applications in the field of bioinformatics such as sequence alignment help to identify the pattern recognition.

Protein Structure Tessellations

Majid Masso discusses Tessellation of Protein Structure by the C_α coordinates in 3-dimension using MATLAB. The building blocks of amino acids is having 20 distinct types in nature (A, C, D, E, F, G, H, I, K, L, M, N, P, Q, R, S, T, V, W, Y). Protein represent in the form of atom, backbone ribbon and tessellation. In these he uses every amino acid into a point of C_α coordinates in 3-dimension by using program in MATLAB.

From the C_α coordinate point which is representing each of amino acids having X, Y, Z vertices which help not to overlap in 3-dimension.

```
 Majid Masso, George Mason University, Manassas, VA
 Coordinates of the points (CA atoms) representing each of the N amino acids, eacl
olumn vector is N-dimensional
=[];y=[];z=[];
 CA trace
lot3(x,y,z);
 Overlap graphs
old on;
 Or just CA points
lot3(x,y,z,'.');
 No axes
xis off;
 Concatenate, w is an Nx3 matrix, each row gives the 3D coordinates of one CA
oint, each CA point is indexed by its row number in w, from 1 to N
=[x(:) y(:) z(:)];
 Tessellation, T is an rx4 matrix, r is the total number of tetrahedral simplices
n the tessellation, the 4 numbers in each row are the indices (row numbers in w)
he CA points forming the vertices of a tetrahedral simplex
=delaunay3(x,y,z);
 Full tessellation (convex hull of tetrahedral simplices)
 FaceAlpha is the the transparency of the triangular faces (set to 0)
etramesh(T,w,'FaceAlpha',0);
 Alternative is tessellation subject to a 12.0 A edge-length cutoff
=size(T);
=s(1,1);
=1;
or i=1:r
e=T(i,1);f=T(i,2);g=T(i,3);h=T(i,4);
if(sqrt((w(e,1)-w(f,1))^2+(w(e,2)-w(f,2))^2+(w(e,3)-w(f,3))^2) <=12.0 &&
qrt((w(e,1)-w(g,1))^2+(w(e,2)-w(g,2))^2+(w(e,3)-w(g,3))^2)<=12.0 &&
qrt((w(e,1)-w(h,1))^2+(w(e,2)-w(h,2))^2+(w(e,3)-w(h,3))^2)<=12.0 &&
qrt((w(f,1)-w(g,1))^2+(w(f,2)-w(g,2))^2+(w(f,3)-w(g,3))^2)<=12.0 &&
qrt((w(f,1)-w(h,1))^2+(w(f,2)-w(h,2))^2+(w(f,3)-w(h,3))^2)<=12.0 &&
qrt((w(g,1)-w(h,1))^2+(w(g,2)-w(h,2))^2+(w(g,3)-w(h,3))^2)<=12.0)
 t(k,1)=e;t(k,2)=f;t(k,3)=g;t(k,4)=h;
 k=k+1;
end
nd
etramesh(t,w,'FaceAlpha',0);
```

fig. 42.3: Delaunay Tesellation in MATLAB

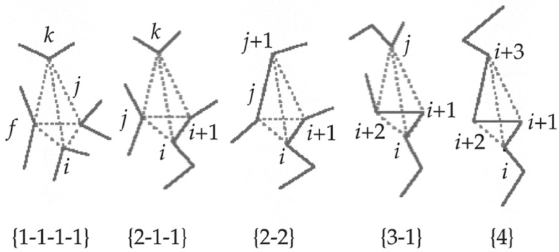


fig. 42.4: Five simplex categories for coordinates

From the above five simplex categories (*fig*. 42.4) distinct tetrahedral and volume for an HIV-1.

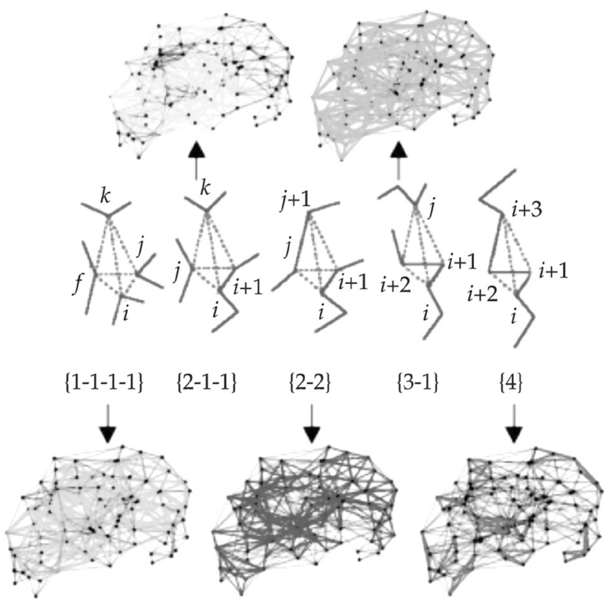


fig. 42.5: Simplex Categories Example

For {1-1-1-1} – 73 simplices mean $T = 0.11$ mean $V = 41.51$

For {2-1-1} – 95 simplices mean $T = 0.18$ mean $V = 19.27$

For {2-2} – 89 simplices mean $T = 0.15$ mean $V = 9.45$

For {3-1} – 109 simplices mean $T = 0.20$ mean $V = 10.09$

For {4} – 16 simplices mean $T = 0.18$ mean $V = 5.61$

Amount of Blood Plasma Protein Adsorption on Nanoparticles

Asavari Mehta used a mathematical model that tells an amount of plasma protein that can adsorb on a particle of carrier which is coated with a Poly ethylene glycol (PEG) that gave the benefit of selection of optimal values

1. PEG molecular weight,
2. PEG mass fraction, and
3. carrier particle diameter which is essential to the creation of a PEG-coated.

The formula obtained from the paper of researcher Gref et al. (2000), in that the Surface density threshold (STD) which represents the small area between PEG chains over the surface of a nanoparticle which creates the blockage of adsorption. The STD formula which contains all the three factors mentioned above, which is used for the representation of the amount of PEG in the form of molecular weight and mass fraction which minimizes protein adsorption on nanoparticles.The estimation of protein adsorption is restricted to the parameter that indicates either the PEG molecular weight or the PEG mass fraction. The protein adsorption due to PEG molecular weight was not valid for PEG mass fraction and vice versa. The correlation coefficient for the correlation test for PEG molecular weight is $R_2 = 0.997$ and for PEG mass fraction is $R_2 = 0.988$. The scheme was not successful in the protein adsorption value that is for the average diameter of the nanoparticles, because there was no experimental detail for nanoparticle diameter data from Gref et al.

So the above-mentioned PEG_{mw} and PEG_{mf} are not accounting for changes in particle diameter. The attempts were to combine the two separate models of PEG_{mw} and PEG_{mf} to find a universal metric that could establish the parameter of a nanoparticle that minimizes the adsorption of protein. One metric was to create a ratio of the two parameters and another was to create the product (multiplication) of the two parameters and it develops a curve for each metric. The results did not give a promise because the variation of each metric which does not produce a matched correlation. Finally, researchers use this modelling tool as a starting point to design PEG-coated, drug-carrier nanoparticles as it related to variation of PEG molecular weight or PEG mass fraction. This model presented in this study for simulating plasma protein adsorption on nanoparticles that would notably inform the fabrication of effective, immuno-deceptive, drug-eluting nanoparticles for cancer treatments.

Conclusion

This is a simple and an easy method in proteins study using MATLAB. While it is simple, it proves its efficiency for protein sequence alignment. A protein study in this MATLAB acts as a good tool which is used to identify the simplest way to align protein sequence.

References

Gref, R., M. Lück, P. Quellec, M. Marchand, E. Dellacherie, S. Harnisch, R. Müller, 2000, 'Stealth' Corona-Core Nanoparticles Surface Modified by Polythylene Glycol (PEG): Influences of the Corona (PEG Chain Lenght and Surface Density) and of the Core Composition on Phagocytic Uptake and Plasma Protein Adsorption. Colloids and Surface B: *Bointerfaces*, 18(3-4), 301-13, doi:10.1016/S0927-7765(99)00156-3.

Masso, Majid, 2010, *Using Biology to Teach Geometry: Protein Structure Tessellations in Matlab*, Laboratory for Structural Bioinformatics George Mason University. (http://binf.gmu.edu/mmasso/MAA2010.pdf)

Mathur, Meghna and Geetika, 2013, "Multiple Sequence Alignment Using MATLAB", *International Journal of Information and Computation Technology*, Number 6.

Mehta, Asavari, 2014, "Development of a MATLAB® Model for Estimating the Amount of Protein Absorption on Nanoparticles Covered with Poly Ethylene Glycol (PEG)", thesis submitted to the Faculty of Drexel University. (https://idea.library.drexel.edu/islandora/object/idea%3A6502)

Zhang, Wen and MengFKe, 2014, "Protein Encoding: A Matlab Toolbox of Representing or Encoding Protein Sequences as Numerical Vectors for Bioinformatics", *Journal of Chemical and Pharmaceutical Research*, 6(7): 2000-2007.

List of Contributors

Agarkar, Sudhakar C., Professor and Dean, VPM's Acedemy of International Education and Research. email: <sagarkar@vpmthane.org>

Akila, B., Assitant Professor, Kanchi Sri Krishna College, Kanchipuram, Tamil Nadu. email: <bakilavs@gmail.com> (Mob.: 8778260815)

Balaji, P., Assistant Professor, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <pbr1002007@ yahoo.com> (Mob.: 9486082115)

Bandi, Shrinik, Professor, Dept. of Mathematics, & Adviser, IPS Academy, Indore. email: <shrenik.bandi@gmail.com>

Bharathi, K., Assistant Professor, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <03bharathi@ gmail.com> (Mob.: 9894281989)

bhuvaneswari, K., PhD Research Scholar, SASTRA, Deemed to be University, Tanjore, Tamil Nadu. email: <yogeshwarsuri@gmail.com> / <ksbhu@yahoo.com>

Chandrasekaran, P.S., PhD Research Scholar, The Kuppuswami Sastri Research Institute, Chennai – 4. email: <ps.chandrasekaran1945@gmail. com>

Dani, Shrikrishna G., Distinguished Professor, UM-DAE Centre for Excellence in Basic Sciences. email: <shrigodani@cbs.ac.in>

Dhanalakshmi, A., Assistant Professor, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <dhana_ amaresan@yahoo.com> (Mob.: 9500538546)

Divya, Ms., MPhil Scholar, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV

University), Kanchipuram – 631561, Tamil Nadu. email: <guruviji97@gmail.com> (Mob.: 9445204713)

E. Geetha, Assistant Professor, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <geethamuthu06@gmail.com> (Mob.: 9566225809)

Gadgil, Mugdha, Associate Professor, Department of Sanskrit and Prakrit Languages Savitribai Phule Pune University, Pune – 411007. email: <mugdha.GB@gmail.com>

Katre, S.A., Chair Professor, Lokmanya Tilak Chair, Savitribai Phule Pune University, Pune – 411007. email: <sakatre@gmail.com>

Kasthuri, S., MPhil Scholar, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu.

Katre, Shailaja S., Visiting Professor, Tilak Maharashtra Vidyapeeth, Pune. email: <shailajaskatre@gmail.com>

Kavitha, T.N., Assistant Professor, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <tnmaths@gmail.com> (Mob.: 9952115346)

Larani, M., MPhil Scholar, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <geethamuthu06@gmail.com> (Mob.: 9566225809)

Limaye, Medha Shrikant, Joint Secretary, Birhanmumbai Ganit Adhyapak Mandal, Mumbai. email: <medhalimaye@gmail.com>

Mishra, Vinod, Professor of Mathematics, Sant Longowal Institute of Engineering and Technology, Punjab. email: <vinodmishra560@gmail.com>

Murali, S., Asst. Prof., Jyotisha, Madras Sanskrti College, Chennai – 600004. email: <muralibhairav@gmail.com>

Narayanan, N. Anil, Assistant Professor & Program Director (BA-Sanskrit), Chinmaya Vishwavidyapeeth, Ernakulam, Kerala – 682313. email: <anilnrynn@yahoo.com>, <anilnarayaranansanskrit@gmail.com>

Padmapriya, R., Former Asst. Professor, Shri Shankarlal Sundarbai Shashun Jain College, Chennai – 17. email: <priyaraman1970@gmail.com>

Pai, R. Venketeswara, Associate Professor, Indian Institute of Science Education and Research (IISER) Pune. email: <venpai79@gmail.com>

Raja Rajeswari, G., Research Scholar, University of Madras. email: <rajarajeshwari.madhavan@gmail.com>

Rajendran, M., Independent Researcher. email: <umahesh2000@gmail.com>

Ram, Sita Sundar, Secretary, Samskrita Academy, Madras, Chennai – 600004. email: <sita.sundarram@gmail.com>

Ramakalyani, V., Project Consultant, HoMI Project, IIT, Gandhinagar, India, Member, Samskrita Academy Madras. email: <ramakalyani1956@gmail.com>

Ramya, S., Assistant Professor, Mahalakshmi College of Arts and Science, Aathur, Uthiramerur, Tamil Nadu. email: <ramyasrini124@gmail.com> (Mob.: 7708764198)

Rao, K. Srinivasa, Professor, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <raokonda@yahoo.com> (Mob.: 9666695522)

Rupa, K., Professor, Department of Mathematics, Global Academy of Technology, Bangalore – 98. email: <rupak@gat.ac.in>

Sarada Devi, Pattisapu, Retired Lecturer, St. Xavier's College, Vishakhapatnam. email: <saradapattisapu@gmail.com>

Saradha, N., Assistant Professor, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <saaradha@yahoo.com> (Mob.: 9843888520)

Sengamalaselvi, J., Assistant Professor, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <pavisneka@gmail.com> (Mob.: 9445408115)

Shailaja, M., Assistant Professor, Government First Grade College, Vijayanagara, Bangalore. email: <shaila.ac55@yahoo.com>

Shakila, A., Teaching Assistant, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <shakisheriff2010@gmail.com> (Mob.: 9791695542)

Sriram, M.S., President, Prof., K.V. Sarma Research Foundation, Chennai – 600020. email: <sriram.physics@gmail.com>

Shubha, B.S., Independent Researcher. email: <shubhashivaniyer@gmail.com>

Shylaja, B.S., Jawaharlal Nehru Planetarium, Bengaluru – 560001. email: <shylaja.jnp@gmail.com>

Thangam, D. Hannah Jerrin, Project fellow, Department of Theoretical Physics, University of Madras. email: <hannahjerinthangam@gmail.com>

Thomann, Johannes, Visiting Scholar, Institute of Asian and Oriental Studies, University of Zurich. email: <johannes.thomann@aoi.uzh.ch>

Vanaja, V., Assistant Professor, Department of Mathematics, Government First Grade College, Yelahanka, Bengaluru – 560064. email: <vanajaksr94@yahoo.com>

Venugopal, Padmaja, Professor & Head, Department of Mathematics, SJB Institute of Technology, Bangalore. email: <venugopalpadmaja@gmail.com>

Vijayalakshmi, PhD Research Scholar, The Kuppuswami Sastri Research Institute, Chennai – 4. email: <vijags@yahoo.com>

Vijayalakshmi, D., Assistant Professor, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <guruviji97@gmail.com> (Mob.: 9445204713)

Uma, S.K., Professor and Head, Department of Mathematics, Sir M Visvesvaraya Institute of Technology. email: <uma_mca@sirmvit.edu>

Umamahesh, V.M., Research Scholar, Sri Chandrasekharendra Viswa Mahavidyalaya. email: <umahesh2000@gmail.com>

Yamuna, C., MSc Scholar, Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya (SCSVMV University), Kanchipuram – 631561, Tamil Nadu. email: <yamuna13199611@gmail.com> (Mob.: 8608434368)